New control and observation schemes based on Takagi-Sugeno models

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Nouveaux schémas de commande et d'observation basés sur les modèles de Takagi-Sugeno

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Doctoral Thesis

New control and observation schemes based on Takagi-Sugeno models

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Abstract

This thesis addresses the estimation and controller design for continuous-time nonlinear systems. The methodologies developed are based on the Takagi-Sugeno (TS) representation of the nonlinear model via the sector nonlinearity approach. All strategies intend to get more relaxed conditions.

The results presented for controller design are split in two parts. The first part is about standard TS models under control schemes based on: 1) a quadratic Lyapunov function (QLF); 2) a fuzzy Lyapunov function (FLF); 3) a line-integral Lyapunov functions (LILF); 4) a novel non-quadratic Lyapunov functional (NQLF). The second part concerns to TS descriptor models. Two strategies are proposed: 1) within the quadratic framework, conditions based on a general control law and some matrix transformations; 2) an extension to the non-quadratic approach based on a line-integral Lyapunov function (LILF) using non-PDC control law schemes and the Finsler's Lemma; this strategy offers parameter-dependent linear matrix inequality (LMI) conditions instead of bilinear matrix inequality (BMI) constraints for second-order systems.

On the other hand, the problem of the state estimation for nonlinear systems via TS models is also addressed considering: a) the particular case where premise vectors are based on measured variables and b) the general case where premise vectors can be based on unmeasured variables. Several examples have been included to illustrate the applicability of the obtained results.

Résumé

Cette thèse aborde l'estimation et la conception de commande de systèmes non linéaires à temps continu. Les méthodologies développées sont basées sur la représentation Takagi-Sugeno (TS) du modèle non linéaire par l'approche du secteur non-linéarité. Toutes les stratégies ont l'intention d'obtenir des conditions plus détendu.

Les résultats présentés pour la conception de commande sont divisés en deux parties. La première partie est environ sur les modèles TS standard au titre des schémas de commande basés sur: 1) une fonction de Lyapunov quadratique (QLF); 2) une fonction de Lyapunov floue (FLF); 3) une fonction de Lyapunov intégrale de ligne (LILF); 4) un nouveau fonctionnelle de Lyapunov non-quadratique (NQLF). La deuxième partie concerne des modèles TS descripteurs. Deux stratégies sont proposées: 1) dans le cadre quadratique, des conditions basées sur une loi de commande général et quelques transformations de matrices; 2) une extension de l'approche non quadratique basée sur LILF utilisant un schéma de commande non-PDC et le lemme du Finsler; cette stratégie offre conditions sur la forme d'inégalité matricielles linéaires (LMI) dépendant des paramètres au lieu des contraintes sur la forme d'inégalité matricielles bilinéaires (BMI) pour les systèmes de second ordre.

D'autre part, le problème de l'estimation de l'état pour les systèmes non linéaires via modèles TS est également abordé considérant: a) le cas particulier où les vecteurs prémisses sont basées sur les variables mesurées et b) le cas général où les vecteurs prémisse peuvent être basés sur des variables non mesurées. Plusieurs exemples ont été inclus pour illustrer l'applicabilité des résultats obtenus.

Especially dedicated to:

God

My parents: Alfonso and Magda E.

My wife and children: Nubia, Raymundo, and Alessia

All my family

My childhood friends

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CHAPTER 1. Introduction

1.1. Context of the thesis

In the last decades a lot of works about nonlinear analysis and design have been conducted on the basis of exact polytopic representation of nonlinear systems also known as Takagi-Sugeno (TS) models (Takagi and Sugeno, 1985). A TS representation can be obtained from a nonlinear model via linearization in several points of interest (Tanaka and Wang, 2001), or via the sector nonlinearity approach, first proposed in (Kawamoto et al., 1992) and extended by (Ohtake et al., 2001; Taniguchi et al., 2001). Loss of information is the main problem of linearization techniques, which give only an approximation of the nonlinear system, a problem that does not appear in the sector nonlinearity approach. Therefore, the sector nonlinearity approach has been usually applied in order to get a TS model. A TS model is composed of a set of linear models blended together with memberships functions (MFs) which contain the model nonlinearities and hold the convex sum property (Tanaka and Wang, 2001). There are many reasons behind the increasing interest on stability analysis and controller/observer design of nonlinear systems via TS models: (a) they can exactly represent a large family of nonlinear models in a compact set of the state space via the sector nonlinearity approach; (b) its convex structure based on membership functions (MFs) allows linear methods to be "easily" mimicked via the direct Lyapunov method (Tanaka and Wang, 2001); (c) appropriate manipulations altogether with parallel distributed compensation (PDC) as a control law usually lead to conditions in the form of linear matrix inequalities (LMIs), which are efficiently solved via convex optimization techniques (Boyd et al., 1994; Scherer and Weiland, 2000).

The TS-LMI framework was originally based on quadratic Lyapunov functions (QLF) such that several results on stability analysis as well as controller/observer design have been widely addressed (Tanaka and Sugeno, 1992; Wang et al., 1996; Tanaka et al., 1998; Patton et al., 1998; Tanaka and Wang, 2001; Bergsten et al., 2002; Ichalal et al., 2008; Z. Lendek et al., 2010b). Nevertheless, LMI conditions thus derived, though simple, were only sufficient, which means that conservativeness is introduced in the solutions, i.e., if the LMI conditions are unfeasible it does not imply that the original problem has no solution. Three independent sources of conservativeness have been identified: (1) the way MFs are removed from nested convex sums to obtain sufficient LMI conditions, (2) the type of Lyapunov function, and (3) the non-uniqueness of the TS model construction. Therefore, a huge effort has been devoted to reach necessity or, at least, relax sufficiency in order to cast a larger family of problems into the TS-LMI framework (Sala et al., 2005; Feng et al., 2005). A lot of results are available that cover partially one or several of these three problems.

For (1), obtaining LMI expressions from nested convex sums has been tackled via matrix properties (Tanaka and Sugeno, 1992; Tuan et al., 2001), via parameter-dependent asymptotically sufficient and necessary conditions (Sala and Ariño, 2007), using triangulation approach to go to asymptotically exact conditions (Kruszewski et al., 2009), and adding slack variables (Kim and Lee, 2000; Liu and Zhang, 2003).

For (2), an important literature is now available that exploit the use of more general Lyapunov function such as piecewise (PWLF) (Johansson et al., 1999; Feng et al., 2005; Campos et al., 2013), fuzzy (FLF, also known as non-quadratic or basis-dependent in the literature) (Tanaka et al., 2003; Guerra and Vermeiren, 2004), and line-integral (LILF) (Rhee and Won, 2006; Mozelli et al., 2009). These general Lyapunov functions share the same MFs than the TS model. The use of PWLF have proved to be particularly difficult to deal with since piecewise generalizations of the quadratic Lyapunov function require extra conditions to guarantee its continuity (Johansson et al., 1999). In the continuous-time framework, fuzzy Lyapunov functions have not met the development of the discrete-time domain (Guerra and Vermeiren, 2004; Guerra et al., 2009; Ding, 2010; Zou and Li, 2011). This asymmetry is due to the fact that the time derivatives of the MFs appear in the analysis and cannot be easily cast as a convex problem; moreover, it leads to local analysis which may create algebraic loops when controller design is concerned (Blanco et al., 2001). Among works on local non-quadratic approach, two directions can be found: those which simply assume a priori known bounds on the time-derivatives of the MFs (Tanaka et al., 2003; Bernal et al., 2006; Mozelli et al., 2009; Zhang and Xie, 2011; Lee et al., 2012; Yoneyama, 2013), and those which rewrite the time-derivative of the MFs as to obtain more structured bounds (Guerra and Bernal, 2009; Bernal and Guerra, 2010; Pan et al., 2012; Jaadari et al., 2012; Lee and Kim, 2014). Should FLFs be used to obtain global conditions, line-integral alternatives can be considered. In the seminal work (Rhee and Won, 2006), the authors showed how line-integral Lyapunov functions can be used to avoid the time derivatives of the MFs at the price of imposing restrictive structures to guarantee the line integral to be path-independent; moreover, this approach leads to bilinear matrix inequalities (BMIs) for controller design; therefore, they are not optimally solvable because existing methods may lead to local minima.

For (3) other convex models besides the TS ones have been used: polynomial (Tanaka et al., 2009), (Sala, 2009) and descriptor (Taniguchi et al., 1999). The descriptor structure appeared in (Luenberger, 1977) with the main interest of describing nonlinear families of systems in a more natural way than the standard state-space one, usually mechanical systems (Luenberger, 1977; Dai, 1989). TS descriptor model is similar to the standard one, the difference is that the descriptor has generally two families of MFs, one for the left-side and the other for the right-side. In (Taniguchi et al., 1999), stability and stabilization of fuzzy descriptor systems have been presented under a quadratic scheme; this work takes advantage of the descriptor structure to reduce the number of LMI constraints, thus reducing the computational burden. Better results for stabilization as well as robust H_∞ controller design have been presented in (Guerra et al., 2007) and (Bouarar et al., 2010) respectively.

The problem of state estimation for dynamical systems is one of the main topics in control theory and has therefore been plentifully treated in the literature; its importance clearly arises from the fact that the control law often depends on state variables which may not be available due to the sensors high cost, inexistence, or impracticality. State estimation both for linear and nonlinear systems have been proposed long ago (Luenberger, 1971; Thau, 1973); more recent works on the subject are: techniques based on sliding mode (Efimov and Fridman, 2011), nonlinear high-gain approach (Khalil and Praly, 2014; Prasov and Khalil, 2013), time-varying gain approach (Farza et al., 2014), and extensions considering unknown inputs are also available (Barbot et al., 2009; Bejarano et al., 2014). Observer design for TS models can be separated in two classes: the first one considers that the MFs depend on measured variables (Tanaka et al., 1998; Patton et al., 1998; Teixeira et al., 2003; Akhenak et al., 2007; Lendek et al., 2010a); the second one assumes that the MFs are also formed by unmeasured variables (Bergsten et al., 2001; Bergsten et al., 2002; Ichalal et al., 2007; Yoneyama, 2009; Lendek et al., 2010a; Ichalal et al., 2011; Ichalal et al., 2012). For the first class, the results obtained in the quadratic framework resemble the characteristic duality observer/controller of linear systems. For the second class, one way to deal with this class of unmeasured variables is to consider extra conditions using classically Lipschitz constants as in (Ichalal et al., 2007). Another way is to use the Differential Mean Value Theorem (DMVT) as in (Ichalal et al., 2011; Ichalal et al., 2012).

It is difficult to extract what are the real important results; there is a need to converge towards the "useful" methods. The ideas followed by this thesis, whatever they are (expanding the Lyapunov function, the control law, the nested sums, the state vector), try to reduce the conservatism of former results. For instance, why is it relevant to introduce control laws whose complexity may lead to less conservative conditions if there are already asymptotic necessary and sufficient (ANS) conditions for quadratic PDC-based controller design? The reason lies on the fact that ANS conditions have been obtained only for convex summations (Sala and Ariño, 2007; Kruszewski et al., 2009) whose computational burden reaches very quickly a prohibitive size for current solvers; thus, approaches preserving asymptotic characteristics while reaching solutions where ANS conditions cannot, are worth exploring. The following example illustrates the limitations of the ANS methods. A TS representation $\dot{x} = \sum_{i=1}^{r} h_i(z)(A_i x + B_i u)$ as well as a PDC control law $u = \sum_{i=1}^{r} h_i(z) F_i x$ are considered in the analysis under a quadratic Lyapunov function $V = x^T P^{-1} x$ (more details in chapter 2). This example is constructed as follows (Delmotte et al., 2007): consider a TS representation with 2 models

$$A_1 = \begin{bmatrix} 0.5 & 0 \\ 1 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & -1 \\ -1 & 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \tag{1.1}$$

that is proved to be stabilizable via an ordinary PDC control law and a quadratic Lyapunov function. Complexity in the representation can be introduced artificially by adding models inside the original polytope. The matrices thus obtained are equally spaced, i.e.: $\left(A_{\delta_k}, B_{\delta_k}\right)$, $\delta_k = \dfrac{k}{r-1}$ with $k \in \{1, 2, \ldots, r-2\}$ corresponds to:

$$A_{1+\delta_k} = (1-\delta_k) A_1 + \delta_k A_2,\ B_{1+\delta_k} = (1-\delta_k) B_1 + \delta_k B_2. \tag{1.2}$$

Thus, the quadratic stabilizability via a PDC control law is guaranteed independently of the number of models r. Stabilization conditions in (Sala and Ariño, 2007) use Polya's property

(Scherer, 2006) introducing extra sums in the initial problem $\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z)h_j(z)\Upsilon_{ij}<0$ with $\Upsilon_{ij}=A_iP+B_iF_j+(*)$, i.e.:

$$\left(\sum_{i=1}^{r}h_i(z)\right)^d\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z)h_j(z)\Upsilon_{ij}<0, \tag{1.3}$$

where d represents the *complexity parameter* for (1.3). Note that if there exists a solution to the initial problem there *must exist* a sufficiently large value of d such that the problem (1.3) is feasible. Theorem 5 of (Sala and Ariño, 2007) also adds some extra variables relaxing the conditions for a fixed value of d. Despite its simplicity, conditions in Theorem 5 of (Sala and Ariño, 2007) with $r=10$ and $d=2$, lead LMI solvers to failure. In this case, the number of LMI conditions and scalar decision variables are 41123 and 772, respectively. This example shows that, sometimes, very simple problems cannot be solved even if ANS conditions are available. Thus, it is important to explore alternatives that provide more relaxed conditions, for instance, non-quadratic Lyapunov functions.

1.2. Scope and objectives

This thesis proposes new schemes of control and observation for TS representations of continuous-time nonlinear systems such that more relaxed conditions are achieved. The problems considered are:

- State feedback controller design.
- Observer design.

The strategies are applied for TS models in a standard or a descriptor form. All developments are based on the Lyapunov's direct method such that LMI conditions (or parameterized ones) are obtained.

1.3. Structure of the thesis

The thesis is organized as follows:

Chapter 2 presents the basis of TS modeling for continuous-time nonlinear systems as well as the main results in the literature about stability analysis, controller/observer design for this sort of TS models under the LMI framework. Additionally, advantages of using descriptor TS representations instead of standard ones are provided.

Chapter 3 provides some strategies on state feedback controller design both for continuous-time standard and descriptor TS models such that less conservative conditions with respect to previous works are achieved. These strategies are based on well-known matrix transformations as well as a variety of Lyapunov functions. Also, a new Lyapunov functional is proposed. In addition, the disturbance rejection problem is addressed.

Chapter 4 considers observer design for continuous-time TS models. Two lines are explored: the particular case where premise vectors are based on measured variables and the general case where some premise variables can be unmeasured. The obtained conditions present better results than those already available in the literature.

Chapter 5 concludes this thesis with final remarks and some future research directions.

1.4. Publications

The main results of my research have been reported or on track to be in the following publications:

International journal publications:

1. T. González, R. Márquez, M. Bernal, and T.M. Guerra. (2015). *Non-quadratic controller and observer design for continuous TS models: a discrete-inspired solution*. International Journal of Fuzzy Systems. DOI: 10.1007/s40815-015-0094-4.
2. R. Márquez, A. Tapia, M. Bernal, and L. Fridman. (2014). *LMI-based second-order sliding set design using reduced order of derivatives*. International Journal of Robust and Nonlinear Control. DOI: 10.1002/rnc.3295.
3. A. Tapia, R. Márquez, M. Bernal, and J. Cortez. (2014). *Sliding subspace design based on linear matrix inequalities*. Kybernetika (50), pp. 436-449.
4. R. Márquez, T.M. Guerra, M. Bernal, and A. Kruszewski. (2015). *A Non-Quadratic Lyapunov Functional for H∞ Control of Nonlinear Systems via Takagi-Sugeno Models*. Journal of The Franklin Institute, minor revision.
5. R. Márquez, T.M. Guerra, M. Bernal, and A. Kruszewski. (2015). A*symptotically Necessary and Sufficient Conditions for Takagi-Sugeno Models using Generalized Non-Quadratic Parameter-Dependent Controller Design*. Fuzzy Set and Systems, first round.

Conference publications:

1. R. Márquez, T.M. Guerra, M. Bernal, and A. Kruszewski. (2015). *Eliminating the parameter-dependence of recent LMI results on controller design of descriptor systems.* In Proceedings of the 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). Istanbul, Turkey. pp 1-6.
2. R. Márquez, T.M. Guerra, M. Bernal, and A. Kruszewski. (2015). *Nested Control Laws for H∞ Disturbance Rejection based on Continuous-Time Takagi-Sugeno*

Models. In Proceedings of 2nd Conference on Embedded Systems, Computational Intelligence and Telematics in Control. Maribor, Slovenia. pp 282-287.

3. R. Márquez, V. Campos, T.M. Guerra, R. Palhares, A. Kruszewski, and M. Bernal. (2014). *H∞ Disturbance Rejection for Continuous-Time Takagi-Sugeno Models based on Nested Convex Sums.* In Proceedings of the Rencontres francophones sur la logique flue et ses applications (LFA), Corsica, France. pp 183-189.
4. R. Márquez, T.M. Guerra, A. Kruszewski, and M. Bernal. (2014). *Decoupled Nested LMI conditions for Takagi-Sugeno Observer Design.* In Proceedings of the 19th IFAC World Congress, Cape Town, South Africa, pp. 7994-7999.
5. R. Márquez, T.M. Guerra, A. Kruszewski, and M. Bernal. (2014). *Non-quadratic Stabilization Of Second Order Continuous Takagi-Sugeno Descriptor Systems via Line-Integral Lyapunov Function*. In Proceedings of the 2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). Beijing, China, pp. 2451-2456.
6. R. Márquez, T.M. Guerra, A. Kruszewski, and M. Bernal. (2013). *Improvements on Non-PDC Controller Design for Takagi-Sugeno Models*. In Proceedings of the 2013 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Hyderabad, India, pp. 1-7.
7. R. Márquez, T.M. Guerra, A. Kruszewski, and M. Bernal. (2013). *Improvements on Non-quadratic Stabilization of Takagi-Sugeno Models via Line-Integral Lyapunov Functions*. In Proceedings of the 3rd IFAC International Conference on Intelligent Control and Automation Science (ICONS), Chengdu, China, pp. 473-478.

CHAPTER 2. Preliminaries on Takagi-Sugeno Models

2.1. Introduction

This chapter presents the basis of modeling under a convex structure of nonlinear systems (TS model) as well as the main results about stability analysis and controller/observer design for this sort of models under quadratic and non-quadratic frameworks. Also, some results on TS models in a descriptor form are provided highlighting the advantages of this scheme when compared to the standard modeling. At the end of chapter some problems to be addressed along this thesis are pointed out.

2.2. Takagi-Sugeno Models

The general form of a nonlinear system is given by

$$\dot{x}(t) = f_1\left(x(t), u(t)\right) \tag{2.1}$$

$$y(t) = f_2\left(x(t), u(t)\right), \tag{2.2}$$

where $x(t) \in \mathbb{R}^{n_x}$ represents the system state vector, $u(t) \in \mathbb{R}^{n_u}$ the input vector, $y(t) \in \mathbb{R}^{n_y}$ the measured output vector, and $f_i(\cdot)$, $i \in \{1,2\}$ are sufficiently smooth nonlinear functions. The state and output equations are defined by (2.1) and (2.2), respectively.

We reduce the family of nonlinear system (2.1)-(2.2) to the affine-in control model:

$$\dot{x}(t) = A\left(x(t)\right)x(t) + B\left(x(t)\right)u(t) \tag{2.3}$$

$$y(t) = C\left(x(t)\right)x(t) + D\left(x(t)\right)u(t), \tag{2.4}$$

where $A(x(t)) \in \mathbb{R}^{n_x \times n_x}$, $B(x(t)) \in \mathbb{R}^{n_x \times n_u}$, $C(x(t)) \in \mathbb{R}^{n_y \times n_x}$, and $D(x(t)) \in \mathbb{R}^{n_y \times n_u}$ are matrices of nonlinear functions.

A nonlinear system can be expressed by the so-called Takagi-Sugeno (TS) model presented in (Takagi and Sugeno, 1985). A TS model is viewed as a convex blending of linear models via membership functions (MFs). The TS modeling is defined by "IF … THEN" rules which represent local linear input-output relations of a nonlinear system.

The R_i fuzzy rules of the TS model are:

IF $z_1(t)$ is $\mathcal{N}_{i1}$ and $z_2(t)$ is $\mathcal{N}_{i2}$ and $\cdots$ $z_p(t)$ is $\mathcal{N}_{ip}$

THEN

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{2.5}$$

$$y(t) = C_i x(t) + D_i u(t), \tag{2.6}$$

where R_i, $i \in \{1,2,\ldots,r\}$, is the i-th rule, r is the number of model rules, $\mathcal{N}_{ij}$, $j \in \{1,2,\ldots,p\}$, are fuzzy sets, $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n_x}$, and $D_i \in \mathbb{R}^{n_y \times n_u}$, $z_1(t), z_2(t), \ldots, z_p(t)$ are the premise variables which may be functions of the states, external disturbances, and/or time. In this thesis, the premise variables are functions of the states (Remark 2.1).

Then, a TS model of a nonlinear system (2.3)-(2.4) can be represented as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(x(t))\right)\left(A_i x(t) + B_i u(t)\right) \tag{2.7}$$

$$y(t) = \sum_{i=1}^{r} h_i\left(z(x(t))\right)\left(C_i x(t) + D_i u(t)\right), \tag{2.8}$$

where $h_i(z(x(t)))$, $i \in \{1,2,\ldots,r\}$, are the membership functions which contain all the nonlinearities and depend on the premise variable vector $z(x(t)) \in \mathbb{R}^p$. The nonlinearities are assumed to be bounded and smooth in a compact set $\mathcal{C}_x$ of the state space which include the desired equilibrium point at $x = 0$. The premise variables vector is formed by all the individual elements $z_1(x(t)), z_2(x(t)), \ldots, z_p(x(t))$ which may depend on measured and/or unmeasured states variables.

Remark 2.1. A more general case is to consider a non affine-in control model, i.e., matrices $A(x(t),u(t))$, $B(x(t),u(t))$, $C(x(t),u(t))$, and $D(x(t),u(t))$ depends also on the control law. However, for TS representations, the MFs are based also on $u(t)$, i.e., $h_i(x(t),u(t))$; it gives algebraic nonlinear equations that can be difficult to solve when dealing with control, it means, $u(t)=\sum_{i=1}^{r} h_i(x(t),u(t))F_i x$. Therefore, only the affine-in control model is considered in this thesis.

If the TS model has a convex sum of nonlinearities in the left-hand side, it may be represented more conveniently via a descriptor TS model (Taniguchi et al., 1999): see section 2.8 for details. Fuzzy polynomial systems preserve the aforementioned structure, but matrices $A_i(x)$, $B_i(x)$, $C_i(x)$, and $D_i(x)$ are matrices of polynomials; consequently, membership functions might be involved polynomials instead of constants altogether nonlinearities (Sala, 2009).

Notation: In the following, for a symmetric matrix M, $M>0$ (resp. $M<0$) means that M is positive definite (resp. negative definite); an asterisk $(*)$ for inline expressions will denote the transpose of the terms on its left-hand side; for matrix expressions, an asterisk will denote the transpose of its symmetric block-entry. When convenient, arguments will be omitted.

The shorthand notation for expressions involving convex sums in Table 2.1 will be adopted whenever considered appropriate.

Table 2.1. Notation for convex sums

Description	Notation
Single convex sum	$\Upsilon_h=\sum_{i=1}^{r} h_i(z)\Upsilon_i$
Double convex sum	$\Upsilon_{hh}=\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Upsilon_{ij}$
"q" nested convex sum	$\Upsilon_{\underbrace{hh\cdots h}_{q}}=\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}(z)h_{i_2}(z)\cdots h_{i_q}(z)\Upsilon_{i_1 i_2\cdots i_q}$
Inverse of a convex sum	$\Upsilon_h^{-1}=\left(\sum_{i=1}^{r} h_i(z(t))\Upsilon_i\right)^{-1}$
Time-derivative of a convex sum	$\Upsilon_{\dot{h}}=\frac{d}{dt}\left(\sum_{i=1}^{r} h_i(z)\Upsilon_i\right)$

2.2.1. Obtaining Takagi-Sugeno Models

The TS representation can be obtained via: 1) linearization in several points of the nonlinear system (Tanaka and Wang, 2001), or 2) sector nonlinearity approach, first proposed in (Kawamoto et al., 1992) and extended by (Ohtake et al., 2001; Taniguchi et al., 2001). Loss of information is the main problem of linearization techniques, a problem that does not appear in the sector nonlinearity approach since it leads to algebraically equivalent representations. Linearization methods as well as polynomial representations are out of the scope of this thesis.

The sector nonlinearity approach consists in the following steps:

1. Identify the nonlinearities $\zeta_j(z(x)) \in \left[\underline{\zeta_j}, \overline{\zeta_j}\right]$ where $\left\{\zeta_j(z(x)), j \in \{1,2,\ldots,p\}\right\}$ is the set of state-dependent non-constant entries in functions $A(x)$, $B(x)$, $C(x)$, $D(x)$ for (2.3)-(2.4), $\underline{\zeta_j}$ and $\overline{\zeta_j}$ are the minimum and maximum bound of $\zeta_j(z(x))$, respectively, in a predefined compact set $\mathcal{C}_x$ that contains the origin.
2. Construct the weighting functions (WFs) in the following form:

$$\omega_0^j(z) = \frac{\overline{\zeta_j} - \zeta_j(z)}{\overline{\zeta_j} - \underline{\zeta_j}}, \quad \omega_1^j(z) = 1 - \omega_0^j(z), \; j \in \{1,2,\cdots,p\}. \tag{2.9}$$

3. Set the membership functions (MFs) as follows:

$$h_i(\cdot) = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}}(\cdot) = \prod_{j=1}^{p} \omega_{i_j}^j(z_j), \; i \in \{1,2,\cdots,2^p\}, \; i_j \in \{0,1\}. \tag{2.10}$$

4. Obtain matrices at the polytope vertex $h_i(\cdot)=1$: $A_i = A(z(\cdot))\big|_{h_i(\cdot)=1}$, $B_i = B(z(\cdot))\big|_{h_i(\cdot)=1}$, $C_i = C(z(\cdot))\big|_{h_i(\cdot)=1}$, $D_i = D(z(\cdot))\big|_{h_i(\cdot)=1}$, $i \in \{1,2,\cdots,r\}$ with $r = 2^p \in \mathbb{N}$.

The MFs satisfy the convex sum property in $\mathcal{C}_x$ due to the way they are constructed, i.e. $\sum_{i=1}^{r} h_i(\cdot) = 1$ and $h_i(\cdot) \geq 0$.

Based on the previous steps, the nonlinear model in (2.3)-(2.4) can be exactly represented in $\mathcal{C}_x$ by the following continuous-time TS model:

$$\dot{x} = \sum_{i=1}^{r} h_i(z)(A_i x + B_i u) = A_h x + B_h u \tag{2.11}$$

$$y = \sum_{i=1}^{r} h_i(z)(C_i x + D_i u) = C_h x + D_h u . \tag{2.12}$$

Remark 2.2. Notice that the TS representation of a nonlinear system via the sector nonlinearity approach is not unique and it is based on the selection of the premise variables. Also, the number of linear models depends on the number of nonlinearities p and increases in an exponential way; hence the importance of selecting the minimum number of nonlinearities as premise variables such that a TS representation remains numerically useful for design purposes which might imply not optimal conditions.

Remark 2.3. Generally a TS model is a *local* representation of the nonlinear system in the compact set of the state space $\mathcal{C}_x = \{x : |x| \le c\}$. However, a *global* model can be obtained if the compact set represents all the state space: $\mathcal{C}_x \in \mathbb{R}^{n_x}$.

The following example shows how the sector nonlinearity approach is used to obtain a TS representation of a nonlinear model. Moreover, an alternative representation of the same model is presented.

Example 2.1. Consider the following nonlinear model:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \sin(x_1) \\ \dot{x}_2 &= x_1 x_2^2 - 3x_2 + (2 + \sin(x_1))u, \end{aligned} \tag{2.13}$$

which can be rewritten as:

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & \sin(x_1) \\ x_2^2 & -3 \end{bmatrix}}_{A(x(t))} x + \underbrace{\begin{bmatrix} 0 \\ 2 + \sin(x_1) \end{bmatrix}}_{B(x(t))} u , \tag{2.14}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

Following the methodology of sector nonlinearity, we have:

1. In (2.14) two nonlinear terms appear: $\zeta_1 = \sin(x_1)$ and $\zeta_2 = x_2^2$; then $p = 2$ and the premise variables are $z_1 = x_1$ and $z_2 = x_2^2$. Assuming, for simplicity, the compact set $\mathcal{C}_x = \{x \in \mathbb{R}^2, |x_2| \le 1\}$; then $\zeta_1 \in [-1,1]$ and $\zeta_2 \in [0,1]$.

2. The construction of WFs yields:

$$\omega_0^1(z) = \frac{1-\sin(z_1)}{2}; \ \omega_1^1(z) = \frac{1+\sin(z_1)}{2};$$

$$\omega_0^2(z) = 1 - z_2; \ \omega_1^1(z) = z_2.$$

3. The MFs obtained are:

$$h_1(z) = \omega_0^1(z)\omega_0^2(z); \ h_2(z) = \omega_1^1(z)\omega_0^2(z);$$

$$h_3(z) = \omega_0^1(z)\omega_1^2(z); \ h_4(z) = \omega_1^1(z)\omega_1^2(z).$$

4. The methodology leads to the following linear matrices ($r = 4$):

$$A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -3 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, \ A_3 = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}, \ A_4 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix},$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_2 = B_4 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Finally, the nonlinear system (2.13) is exactly represented by the TS model (2.11) in the compact set of the state space $\mathcal{C}_x$.

Now, in order to show the non-uniqueness of the TS model, if (2.13) is rewritten as:

$$\dot{x} = \begin{bmatrix} -1 & \sin(x_1) \\ 0 & -3 + x_1 x_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 + \sin(x_1) \end{bmatrix} u. \tag{2.15}$$

Note that in this case, it is necessary to "know" something about x_1, for example using either $\mathcal{C}_x = \{x \in \mathbb{R}^2, |x_1| \le \pi, |x_2| \le 1\}$ or directly $\mathcal{C}_x = \{x \in \mathbb{R}^2, |x_1 x_2| \le \pi\}$. Even if not equivalent using either the first or the second compact set $\mathcal{C}_x$ will give the same matrices. Following the sector nonlinearity approach, an alternative of TS model of the form (2.11) can be obtained, where: $A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -3-\pi \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -3-\pi \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & -1 \\ 0 & -3+\pi \end{bmatrix}$, $A_4 = \begin{bmatrix} -1 & 1 \\ 0 & -3+\pi \end{bmatrix}$, $B_1 = B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B_2 = B_4 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\zeta_1 = \sin(x_1)$, $\zeta_2 = x_1 x_2$, $z_1 = x_1$, $z_2 = x_1 x_2$, $\zeta_1 \in [-1,1]$, $\zeta_2 \in [-\pi, \pi]$, $\omega_0^1(z) = \frac{1-\sin(z_1)}{2}$, $\omega_1^1(z) = \frac{1+\sin(z_1)}{2}$, $\omega_0^2(z) = \frac{\pi - z_2}{2\pi}$, $\omega_1^1(z) = \frac{\pi + z_2}{2\pi}$, $h_1(z) = \omega_0^1(z)\omega_0^2(z)$, $h_2(z) = \omega_1^1(z_1)\omega_0^2(z_2)$, $h_3(z) = \omega_0^1(z)\omega_1^2(z)$, $h_4(z) = \omega_1^1(z)\omega_1^2(z)$. In this example, both representations are *local* with respect to the nonlinear system because either " x_2 " or " $x_1 x_2$ " are bounded.♦

2.3. Lyapunov functions

The stability analysis and control design of TS models are based on the direct Lyapunov method (DLM) (Tanaka and Sugeno, 1992), which requires a Lyapunov function candidate (normally quadratic) to be proposed in order to find sufficient conditions to ensure the system trajectories to be asymptotically driven to the origin. Details about Lyapunov stability are given in Appendix A whereas this section analyzes several structures of Lyapunov function candidates that have been proposed in the literature in order to find conditions such that stability/stabilization of TS models is guaranteed. Without loss of generality, in the rest of this thesis, the considered equilibrium point for stability is supposed to be at $x = 0$.

2.3.1. Quadratic Lyapunov function

The most popular is the *quadratic* Lyapunov function (QLF) which has the following form:

$$V(x) = x^T P x, \tag{2.16}$$

or

$$V(x) = x^T P^{-1} x, \tag{2.17}$$

where $P \in \mathbb{R}^{n_x \times n_x}$ is a symmetric and positive definite matrix.

However, only sufficient conditions are derived when a common QLF is used to solve the stability/stabilization problem, which means that conservativeness is introduced in the solutions. To tackle this inconvenience, new structures for the Lyapunov function candidate have been proposed; for instance: fuzzy (also known as non-quadratic) (Tanaka et al., 2003; Guerra and Vermeiren, 2004), line-integral (Rhee and Won, 2006; Mozelli et al., 2009), piecewise (Johansson et al., 1999; Feng et al., 2005; Campos et al., 2013), and polynomial (Prajna et al., 2004). Piecewise and polynomial Lyapunov functions are not considered in this thesis.

2.3.2. Fuzzy Lyapunov function

The *fuzzy* Lyapunov function (FLF) is given by:

$$V(x) = x^T \left(\sum_{i=1}^{r} h_i(z) P_i \right) x = x^T P_h x, \tag{2.18}$$

or

$$V(x) = x^T \left(\sum_{i=1}^{r} h_i(z) P_i \right)^{-1} x = x^T P_h^{-1} x, \tag{2.19}$$

with $P_i \in \mathbb{R}^{n_x \times n_x}$, $i \in \{1, 2, \cdots, r\}$ as symmetric positive-definite matrices. The FLF shares the same MFs $h_i(z)$ of the TS model; they of course satisfy the convex-sum property. This sort of Lyapunov function reduces conservativeness and quadratic results are a particular case of it; both the continuous- and discrete-time domain have seen important improvements, though the continuous-time case remain more complicated since the time derivatives of the MFs will appear in the analysis.

2.3.3. Line-Integral Lyapunov function

The *line-integral* Lyapunov function (LILF) candidate has the next structure (Khalil, 2002):

$$V(x) = 2 \int_{\Gamma(0,x)} \mathfrak{F}(\psi) d\psi, \tag{2.20}$$

where $\Gamma(0, x)$ is any path from the origin to the current state x, $\psi \in \mathbb{R}^{n_x}$ is a dummy vector for the integral, $d\psi \in \mathbb{R}^{n_x}$ is an infinitesimal displacement vector.

In order to have (2.20) well defined, integral part has to be path independent, i.e., $\mathfrak{F}(\psi)$ has to be the gradient of a continuous positive function of x; it is satisfied through the conditions in the following lemma, more details in (Khalil, 2002).

Lemma 2.1. (path-independency): Let $\mathfrak{F}(x) = \left[\mathfrak{F}_1(x), \mathfrak{F}_2(x), \ldots, \mathfrak{F}_{n_x}(x) \right]^T$. A necessary and sufficient condition for $V(x)$ to be a path-independent function is

$$\frac{\partial \mathfrak{F}_i(x)}{\partial x_j} = \frac{\partial \mathfrak{F}_j(x)}{\partial x_i}, \tag{2.21}$$

for $i, j \in \{1, 2, \cdots, n_x\}$.

Proof: The condition above is the condition for a line-integral to be path-independent (Khalil, 2002). A special structure on $\mathfrak{F}(x)$ has been proposed in (Rhee and Won, 2006) such as the path-independency condition is satisfied:

$$\mathfrak{F}(x) = \left(\sum_{i=1}^{r} h_i(x)\left(\bar{P} + \mathfrak{D}_i\right) \right) x = P(x)x, \tag{2.22}$$

with

$$\bar{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n_x} \\ p_{12} & 0 & \cdots & p_{2n_x} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n_x} & p_{2n_x} & \cdots & 0 \end{bmatrix}, \quad \mathfrak{D}_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n_x n_x}^{\alpha_{in_x}} \end{bmatrix}, \quad h_i(x) = \prod_{j=1}^{n_x} \omega_j^{\alpha_{ij}}(x_j) \quad \text{where}$$

$\omega_j^{\alpha_{ij}}(x_j)$ are the WFs, and $\bar{P} + \mathfrak{D}_i = \left(\bar{P} + \mathfrak{D}_i\right)^T > 0$.

Despite the fact that this structure avoids the appearance of time derivatives of the MFs leading to global conditions to the problem of stability/stabilization for TS models within a compact set $\mathcal{C}_x$, the special structure on the MFs ($\omega_1^{\alpha_{i1}}(x_1)$, $\omega_2^{\alpha_{i2}}(x_2), \ldots, \omega_{n_x}^{\alpha_{in_x}}(x_{n_x})$) cannot be easily satisfied; for instance, consider the following nonlinear model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \sin x_2 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.23}$$

It is clear that the MFs depend on both states because of the term "$x_1 \sin x_2$"; therefore, the special structure above cannot be satisfied.

2.4. Stability Analysis of Takagi-Sugeno Models

Stability analysis of TS models consists in deriving sufficient conditions to guarantee the stability of a nonlinear system in the TS form; these conditions are preferably written as *linear matrix inequalities* (LMI) because they are efficiently solved via convex optimization techniques (Boyd et al., 1994; Scherer and Weiland, 2000); see Appendix B for more details on LMI problems as well as some properties. Expressing conditions in terms of LMIs is not a trivial task. In some approaches, multiple convex sums appear; then, in order to obtain LMI conditions from multiple-summation negativity problem, different relaxations have been developed which help to drop off the MFs from nested convex sums. All relaxations considered in this thesis are presented in Appendix C.

Some works have been developed in this direction. These works can be separated depending on the Lyapunov function they are based on: *quadratic* (Tanaka and Wang, 2001) and *non-quadratic* (Blanco et al., 2001; Tanaka et al., 2003; Rhee and Won, 2006; Guerra and Bernal, 2009; Mozelli et al., 2009; Bernal and Guerra, 2010). Some recalls about them are given below.

2.4.1. Quadratic Stability of TS Models

Consider the TS model (2.11) without inputs ($u = 0$), that writes:

$$\dot{x} = \sum_{i=1}^{r} h_i(z) A_i x = A_h x, \tag{2.24}$$

and taking into account the QLF as in (2.16), condition $\dot{V}(x) < 0$ is satisfied if:

$$\sum_{i=1}^{r} h_i(z)\left(PA_i + A_i^T P\right) < 0. \tag{2.25}$$

The following result has been presented in (Tanaka and Wang, 2001) and it provides conditions to guarantee stability of the origin in the TS model (2.24).

Theorem 2.1. The equilibrium point of TS model (2.24) is globally asymptotically stable if there exists a common symmetric positive-definite matrix P such that the following conditions hold:

$$PA_i + A_i^T P < 0, \ \forall i \in \{1, 2, \ldots, r\}. \tag{2.26}$$

Remark 2.4. The stability conditions above are only sufficient because no information about the MFs (h_i) is considered. This fact, altogether with the requisite of finding a common matrix P for all subsystems A_i, brings a strong problem of conservativeness, i.e., if the problem is unfeasible it does not imply that the nonlinear model is not stable: in other words, there is still room for improvements.

Remark 2.5. If the modeling region $\mathcal{C}_x \subset \mathbb{R}^n$ is not $\mathbb{R}^n$, proving *global* stability of a TS model means proving *local* stability of the original nonlinear system.

Example 2.2. Consider the autonomous nonlinear model:

$$\begin{aligned} \dot{x}_1 &= -2x_1 - x_2 \\ \dot{x}_2 &= x_1 x_2^2 - 0.5 x_2, \end{aligned} \tag{2.27}$$

which can be rearranged as:

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ 0 & -0.5 + x_1 x_2 \end{bmatrix} x, \tag{2.28}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. The nonlinear model (2.28) can be rewritten via sector nonlinearity approach as a TS model (2.24) in the compact set $\mathcal{C}_x = \{x \in \mathbb{R}^2 : |x_1 x_2| \le 1\}$ with the following matrices and MFs:

$A_1 = \begin{bmatrix} -2 & -1 \\ 0 & -1.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & -1 \\ 0 & 0.5 \end{bmatrix}$, $\zeta_1 = x_1 x_2$, $z_1 = x_1 x_2$, $h_1(z) = \omega_0^1(z) = \dfrac{1 - z_1}{2}$, $h_2(z) = \omega_1^1(z) = \dfrac{1 + z_1}{2}$.

Conditions (2.26) fail to find a solution for this problem. However, it does not imply that the nonlinear system is not stable; consider the same nonlinear model (2.27) can be expressed as:

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ x_2^2 & -0.5 \end{bmatrix} x, \tag{2.29}$$

which can be represented as a TS model (2.24) in the same compact set $\mathcal{C}_x = \{x \in \mathbb{R}^2, |x_1| \le 1\}$ with the following matrices and MFs:

$A_1 = \begin{bmatrix} -2 & -1 \\ 0 & -0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & -1 \\ 1 & -0.5 \end{bmatrix}$, $\zeta_1 = x_2^2$, $z_1 = x_2^2$, $h_1(z) = \omega_0^1(z) = 1 - z_1$, $h_2(z) = \omega_1^1(z) = z_1$. A feasible solution is achieved with $P = \begin{bmatrix} 0.3690 & 0.0594 \\ 0.0594 & 0.8535 \end{bmatrix}$.♦

The following section describes alternatives to reduce the conservativeness of quadratic solutions by using different non-quadratic Lyapunov functions for stability analysis of TS models.

2.4.2. Non-Quadratic Stability of TS Models

Consider the FLF (2.18). Its time-derivative along the trajectories of the TS model (2.24) is:

$$\dot{V}(x) = x^T \left(P_h A_h + A_h^T P_h + P_{\dot{h}} \right) x. \tag{2.30}$$

Condition $\dot{V}(x) < 0$ holds if:

$$P_h A_h + A_h^T P_h + P_{\dot{h}} < 0, \tag{2.31}$$

which is equivalent to:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \left(P_j A_i + A_i^T P_j + \sum_{k=1}^{r} \dot{h}_k(z) P_k \right) < 0. \tag{2.32}$$

It is important to notice from (2.32) that the term $P_{\dot{h}} = \sum_{k=1}^{r} \dot{h}_k(z) P_k$ implies a dependency on the time derivatives of the MFs, which are difficult to cast as a convex problem.

A first approach for dealing with this term was presented in (Tanaka et al., 2003), using a direct bound on the time derivatives of MFs in the following way:

$$P_{\dot{h}} = \sum_{k=1}^{r} \phi_k P_k \,, \ \left| \dot{h}_k(z) \right| \le \phi_k \,. \tag{2.33}$$

Now, considering that

$$\sum_{k=1}^{r} \dot{h}_k(z) = 0 \,, \ \forall z \,, \tag{2.34}$$

can be written as

$$\dot{h}_r(z) = -\sum_{k=1}^{r-1} \dot{h}_k(z), \tag{2.35}$$

which leads to the next theorem.

Theorem 2.2. Assume that $\left| \dot{h}_k(z) \right| \le \phi_k$. The TS model (2.24) is stable if there exist $\phi_k \ge 0$, $k \in \{1,2,\ldots,r-1\}$ such that:

$$P_i = P_i^T > 0 \,, \ \forall i \in \{1,2,\ldots,r\} \tag{2.36}$$

$$P_k \ge P_r \,, \ \forall k \in \{1,2,\ldots,r-1\} \tag{2.37}$$

$$\frac{1}{2}\left(P_j A_i + A_i^T P_j + P_i A_j + A_j^T P_i \right) + \sum_{k=1}^{r-1} \phi_k \left(P_k - P_r \right) < 0, \ \forall (i,j) \in \{1,2,\ldots,r\}^2 \,, \ i \le j \,. \tag{2.38}$$

The approaches for non-quadratic stability presented so far try, in a sense, to give a "global" stability result within a compact set $\mathcal{C}_x$ for nonlinear systems in a TS form. Unfortunately, global stability cannot be achieved for many nonlinear systems. The next proposal, presented in (Guerra and Bernal, 2009), is concerned with deriving local stability conditions instead of global ones, an idea which matches nonlinear analysis and design for models that do not admit global solutions (Khalil, 2002).

In (Guerra and Bernal, 2009) the way to deal with $P_{\dot{h}}$ is using the MFs' information to find local conditions through the following relation:

$$P_{\dot{h}} = \sum_{i=1}^{r}\sum_{k=1}^{p} h_i \frac{\partial w_0^k}{\partial z_k}\left(P_{g_1(i,k)} - P_{g_2(i,k)}\right)\dot{z}_k , \tag{2.39}$$

with $g_1(i,k) = \lfloor (i-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (i-1) \bmod 2^{p-k}$ and $g_2(i,k) = g_1(i,k) + 2^{p-k}$, $\lfloor \bullet \rfloor$ stands for the floor function. They also consider the premise vector as $z(x) = Lx$ with $L \in \mathbb{R}^{p \times n_x}$, which entails a linear combination of the states. Then, the time-derivative of the premise vector yields

$$\dot{z}_k = \sum_{l=1}^{n_x}\left(LA_h\right)_{kl} x_l = \sum_{j=1}^{r}\sum_{l=1}^{n_x} h_j \left(LA_j\right)_{kl} x_l , \tag{2.40}$$

where $\left(LA_j\right)_{kl}$ represents the k-ith row and l-ith column entry of LA_j. After substitution in (2.39):

$$P_{\dot{h}} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{p}\sum_{l=1}^{n_x} h_i h_j \frac{\partial w_0^k}{\partial z_k} x_l \left(LA_j\right)_{kl}\left(P_{g_1(i,k)} - P_{g_2(i,k)}\right). \tag{2.41}$$

Recall that the $\dot{V}(x) < 0$ holds if:

$$P_h A_h + A_h^T P_h + P_{\dot{h}} < 0 , \tag{2.42}$$

which, taking into account (2.41), the bound $\left|\frac{\partial w_0^k}{\partial z_k} x_l\right| \le \lambda_{kl}$ for $k \in \{1,2,\ldots,p\}$, $l \in \{1,2,\ldots,n_x\}$, and all the possible sign combinations (property B.6: because unsigned term $P_{\dot{h}}$), can be rewritten as:

$$P_h A_h + A_h^T P_h + \sum_{k=1}^{p}\sum_{l=1}^{n_x}(-1)^{d_{kl}^m} \lambda_{kl}\left(LA_h\right)_{kl}\left(P_{g_1(h,k)} - P_{g_2(h,k)}\right) < 0 , \tag{2.43}$$

which leads to the following theorem:

Theorem 2.3. The equilibrium point of TS model (2.24) is locally asymptotically stable in the outermost Lyapunov level $\mathcal{D} = \left\{x : x^T P_h x \le c\right\}$ contained in the compact set $\mathcal{C}_x$ and

$\bar{\mathcal{D}} = \bigcap_{k,l} \left\{ x : \left| \frac{\partial w_0^k}{\partial z_k} x_l \right| \le \lambda_{kl} \right\}$, if there exist symmetric positive definite matrices P_i, $i \in \{1, 2, \ldots, r\}$, such that the following conditions hold:

$$\begin{aligned} & \Upsilon_{ii}^m < 0,\ \forall i \in \{1,2,\ldots,r\},\ \forall m \in \left\{1,2,\ldots,2^{p \times n_x}\right\} \\ & \frac{2}{r-1}\Upsilon_{ii}^m + \Upsilon_{ij}^m + \Upsilon_{ji}^m < 0,\ \forall (i,j) \in \{1,2,\ldots,r\}^2,\ i \ne j,\ \forall m \in \left\{1,2,\ldots,2^{p \times n_x}\right\}, \end{aligned} \tag{2.44}$$

with $\Upsilon_{ij}^m = P_i A_j + A_j^T P_i + \sum_{k=1}^{p} \sum_{l=1}^{n_x} (-1)^{d_{kl}^m} \lambda_{kl} \left(LA_j\right)_{kl} \left(P_{g_1(i,k)} - P_{g_2(i,k)}\right)$, d_{kl}^m defined from the binary representation of $m-1 = d_{pn_x}^m + d_{p(n_x-1)}^m \times 2 + \cdots + d_{11}^m \times 2^{p \times (n_x - 1)}$, $g_1(i,k)$ and $g_2(i,k)$ defined as in (2.39).

Remark 2.6. Theorem 2.3 provides *local* conditions for stability problem of nonlinear systems in the Takagi-Sugeno form. They guarantee stability of the TS model (2.24) for the outermost Lyapunov level $\mathcal{D}$ contained in region $\mathcal{C}_x$ and $\bar{\mathcal{D}}$: $\mathcal{D} \subseteq \left(\mathcal{C}_x \cap \bar{\mathcal{D}}\right)$.

Remark 2.7. Conditions in Theorem 2.3 are LMI because the values of bounds $\lambda_{kl} > 0$ can be calculated a priori. If no solution is achieved with these bounds, the largest region of attraction can be found via a dichotomy search algorithm under the assumption that the problem $P_h A_h + A_h^T P_h < 0$ has a solution.

Remark 2.8. It is clear that conditions in Theorem 2.3 include the *quadratic stability* as a particular case since $P_i = P$ means $P_{g_1(i,k)} - P_{g_2(i,k)} = 0$ and conditions (2.44) are exactly (2.26), i.e. $PA_i + A_i^T P < 0$.

An alternative has been presented in (Sala et al., 2010) in order to consider a nonlinear structure in the premise vector $z(x)$ instead of a linear one. Then, the term $\dot{P}_h$ is written as:

$$\begin{aligned} \dot{P}_h &= \sum_{i=1}^{r} \sum_{k=1}^{p} h_i \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(i,k)} - P_{g_2(i,k)}\right) \dot{z}_k \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{p} \sum_{l=1}^{n_x} \sum_{s=1}^{n_x} h_i h_j \frac{\partial w_0^k}{\partial x_s} x_l \left(A_j\right)_{sl} \left(P_{g_1(i,k)} - P_{g_2(i,k)}\right), \end{aligned} \tag{2.45}$$

with $g_1(i,k)$ and $g_2(i,k)$ defined as in (2.39). After substitution of (2.45) in (2.42) and assuming the bound $\left|\frac{\partial w_0^k}{\partial x_s} x_l\right| \le \lambda_{ksl}$ for $k \in \{1,2,\dots,p\}$, $l,s \in \{1,2,\dots,n_x\}$, and all the possible sign combinations (property B.6: because unsigned term $\dot{P}_h$), the conditions are given by:

$$P_h A_h + A_h^T P_h + \sum_{k=1}^{p}\sum_{l=1}^{n_x}\sum_{s=1}^{n_x}(-1)^{d_{kls}^m} \lambda_{kls} (A_h)_{sl} \left(P_{g_1(h,k)} - P_{g_2(h,k)}\right) < 0. \tag{2.46}$$

Theorem 2.4. The equilibrium point of TS model (2.24) is locally asymptotically stable in the outermost Lyapunov level $\mathcal{D} = \{x : x^T P_h x \le c\}$ contained in the compact set $\mathcal{C}_x$ and $\bar{\mathcal{D}} = \bigcap_{k,l,s} \left\{x : \left|\frac{\partial w_0^k}{\partial x_s} x_l\right| \le \lambda_{kls}\right\}$, if there exist symmetric positive definite matrices P_i, $i \in \{1,2,\dots,r\}$, such that the following conditions hold:

$$\begin{array}{l} \Upsilon_{ii}^m < 0,\ \forall i \in \{1,2,\dots,r\},\ \forall m \in \left\{1,2,\dots,2^{p\times n_x^2}\right\} \\ \dfrac{2}{r-1}\Upsilon_{ii}^m + \Upsilon_{ij}^m + \Upsilon_{ji}^m < 0,\ \forall (i,j) \in \{1,2,\dots,r\}^2,\ i \ne j,\ \forall m \in \left\{1,2,\dots,2^{p\times n_x^2}\right\}, \end{array} \tag{2.47}$$

with $\Upsilon_{ij}^m = P_i A_j + A_j^T P_i + \sum_{k=1}^{p}\sum_{l=1}^{n_x}\sum_{s=1}^{n_x}(-1)^{d_{kls}^m} \lambda_{kls} (A_j)_{sl} \left(P_{g_1(i,k)} - P_{g_2(i,k)}\right)$, d_{kls}^m defined from the binary representation of $m-1 = d_{pn_xn_x}^m + d_{pn_x(n_x-1)}^m \times 2 + \cdots + d_{111}^m \times 2^{pn_x^2-1}$, $g_1(i,k)$ and $g_2(i,k)$ defined as in (2.39).

2.5. Controller Design of Takagi-Sugeno Models

A control law must be designed such that the closed-loop nonlinear system in a TS form is stable. As in the stability problem, the stabilization conditions are written as *linear matrix inequalities* (LMI) and different relaxations can be applied.

Several approaches have been presented in order to solve the stabilization problem. These approaches are based on: *quadratic Lyapunov function* (Wang et al., 1996; Tanaka et al., 1998; Jaadari et al., 2012) and *non-quadratic Lyapunov function* (Tanaka et al., 2003; Rhee and Won, 2006; Mozelli et al., 2009; Bernal et al., 2010; Guerra et al., 2012; Pan et al., 2012; Jaadari et al., 2012).

2.5.1. Control law

In order to deal with the stabilization problem there exist different options of state-feedback control laws in the literature. A classical control law is the parallel distributed compensation (PDC), first proposed in (Wang et al., 1996). The PDC controller is given by

$$u = \sum_{i=1}^{r} h_i(z) F_i x = F_h x, \tag{2.48}$$

where $F_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \{1, 2, \ldots, r\}$.

Introducing directly the inverse matrix for quadratic stabilization gives an equivalent form that is more convenient to obtain LMI constraints problems and discuss the extensions of such control laws, i.e.:

$$u = \sum_{i=1}^{r} h_i(z) F_i P^{-1} x = F_h P^{-1} x. \tag{2.49}$$

The main advantage of this controller is that it shares the same MFs of the TS model; thus, the convex structure allows the direct Lyapunov method to be straightforwardly applied to controller synthesis. Moreover, the controller gains can be calculated through linear matrix inequalities (LMIs).

When the PDC control law (2.49) is substituted in the TS model (2.11), the following closed-loop TS model is obtained:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) \left(A_i + B_i F_j P^{-1} \right) x = \left(A_h + B_h F_h P^{-1} \right) x. \tag{2.50}$$

Another sort of control laws have been stated in (Guerra and Vermeiren, 2004) which are called non-PDC control laws:

$$u = \left(\sum_{i=1}^{r} h_i(z) F_i \right) \left(\sum_{j=1}^{r} h_j(z) P_j \right)^{-1} x = F_h P_h^{-1} x, \tag{2.51}$$

$$u = \left(\sum_{i=1}^{r} h_i(z) F_i \right) \left(\sum_{j=1}^{r} h_j(z) H_j \right)^{-1} x = F_h H_h^{-1} x, \tag{2.52}$$

where $F_i \in \mathbb{R}^{n_u \times n_x}$, $P_j \in \mathbb{R}^{n_x \times n_x}$, and $H_j \in \mathbb{R}^{n_x \times n_x}$ for $i, j \in \{1, 2, \ldots, r\}$. These control laws have been proposed to be used altogether with NQLF for discrete-time TS models. The non-PDC control law (2.51) still has a link with the Lyapunov function through the symmetric and positive-definite matrices P_j. In contrast, (2.52) cuts this link and gives a more relaxed control law by adding the free slack variables H_j.

The following closed-loop TS model are obtained using (2.51) and (2.52), respectively:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \left(A_i + B_i F_j \left(\sum_{k=1}^{r} h_k(z) P_k \right)^{-1} \right) x = \left(A_h + B_h F_h P_h^{-1} \right) x \tag{2.53}$$

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \left(A_i + B_i F_j \left(\sum_{k=1}^{r} h_k(z) H_k \right)^{-1} \right) x = \left(A_h + B_h F_h H_h^{-1} \right) x . \tag{2.54}$$

Remark 2.9. The control laws (2.51) and (2.52) have been applied to continuous-time TS models in (Bernal et al., 2006) and (Jaadari et al., 2012), respectively.

2.5.2. Quadratic stabilization of TS Models

Consider the closed-loop TS model (2.50) and based on the QLF given in (2.17), condition $\dot{V}(x) < 0$ is satisfied if:

$$P^{-1} A_h + A_h^T P^{-1} + P^{-1} B_h F_h + F_h^T B_h^T P^{-1} < 0 . \tag{2.55}$$

Congruence property with P is applied, which leads to the following conditions:

$$A_h P + P A_h^T + B_h F_h + F_h^T B_h^T < 0 . \tag{2.56}$$

The previous conditions can be expressed as a double nested convex sum:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \left(A_i P + P A_i^T + B_i F_j + F_j^T B_i^T \right) < 0 . \tag{2.57}$$

In order to obtain LMI conditions from multiple-summation negativity problems, as in (2.57), different relaxations have been developed which help dropping off the MFs from them. These relaxations are presented in Appendix C. Then, the following stabilization theorem can be formulated:

Theorem 2.5. (Tanaka et al., 1998)**:** The TS model (2.11) under the PDC control law (2.49) is globally asymptotically stable if there exist a symmetric and positive definite matrix P and matrices F_j, $j \in \{1, 2, \ldots, r\}$, such that the following conditions hold:

$$\begin{aligned} &\Upsilon_{ii} < 0, \ \forall i \in \{1, 2, \cdots, r\} \\ &\Upsilon_{ij} + \Upsilon_{ji} < 0, \ \forall (i, j) \in \{1, 2, \cdots, r\}^2, \ i < j, \end{aligned} \tag{2.58}$$

with $\Upsilon_{ij} = A_i P + P A_i^T + B_i F_j + F_j^T B_i^T$.

The next result for quadratic stabilization of TS models using the QLF provided in (2.17), the non-PDC control law (2.52), and the well-known Finsler's lemma, has been developed in (Jaadari et al., 2012).

Theorem 2.6. The TS model (2.11) under the non-PDC control law (2.52) is globally asymptotically stable if $\exists \varepsilon > 0$, a symmetric and positive definite matrix P and matrices F_j, H_j, $j \in \{1, 2, \ldots, r\}$ such that the following conditions hold:

$$\begin{aligned} &\Upsilon_{ii} < 0, \ \forall i \in \{1, 2, \cdots, r\} \\ &\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \ \forall (i,j) \in \{1, 2, \cdots, r\}^2, \ i \neq j, \end{aligned} \tag{2.59}$$

with $\Upsilon_{ij} = \begin{bmatrix} A_i H_j + B_i F_j + (*) & (*) \\ H_j - P + \varepsilon\left(A_i H_j + B_i F_j\right) & -2\varepsilon P \end{bmatrix}$.

Remark 2.10. Conditions in Theorem 2.6 are parameter-dependent LMIs; they are LMI up to the choice of ε. Nevertheless, it has been shown in (de Oliveira and Skelton, 2001) and (Oliveira et al., 2011) that a logarithmically spaced family of values, for instance $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$, is adequate to avoid an exhaustive linear search of feasible solutions.

Remark 2.11. An important feature of TS-LMI framework is the fact that specifications and/or constraints such as decay rate, H_∞ disturbance attenuation, constraint on the input, constraint on the output, etc., can be introduced in a natural way (Zs. Lendek et al., 2010; Tanaka and Wang, 2001).

2.5.3. Non-quadratic stabilization of TS Models

Some of these approaches consider the FLF given in (2.18), its time derivative being

$$\dot{V}(x) = \dot{x}^T P_h x + x^T P_h \dot{x} + x^T P_{\dot{h}} x, \tag{2.60}$$

where $P_{\dot{h}} = \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) P_k$.

As in stability analysis, different strategies have been developed in order to handle $P_{\dot{h}}$ for stabilization; most of them consider bounds on the time-derivatives of MFs ($\dot{h}_k$), despite the fact that there is no direct extension from stability. These approaches are more questionable for stabilization as, by chain rule, the time derivatives of the MFs can contain the control action to be designed, so the validity region of the obtained controller must be checked a posteriori.

In (Tanaka et al., 2003), BMI conditions are obtained, these conditions are rewritten as LMIs after the well-known completion square technique is applied; naturally, these are conservative results. In (Tanaka et al., 2007) a redundancy descriptor system is used to directly get the design conditions in terms of LMIs; moreover, a significant reduction of computational complexity is achieved. Another alternative was presented in (Mozelli et al., 2009), where slack variables are introduced to provide new degrees of freedom to the problem: parameter-dependent LMI conditions are obtained.

Whatever the approach under consideration, they are all based on the assumption that bounds on the time-derivatives of MFs are known. On the other hand, they still use PDC control laws without taking into account the fuzzy structure of the Lyapunov function. The main drawback of these approaches is the fact that generally all the bounds are dependent on the control law u, which is not possible to know beforehand for controller design. This problem is highlighted with the following example:

Example 2.3. Consider the following nonlinear model:

$$\dot{x} = ax + \left(x^3 + b\right)u. \tag{2.61}$$

This model can be turned into the TS form (2.11) for the compact set $\mathcal{C}_x \in \{x : |x| \le d\}$ where $A_1 = A_2 = a$, $B_1 = d^3 + b$, $B_2 = -d^3 + b$, $h_1(x) = w_0^1 = \dfrac{d^3 + x^3}{2d^3}$, $h_2(x) = 1 - w_0^1$. Then, the time derivatives $\dot{h}_1(x)$ and $\dot{h}_2(x)$ are:

$$\dot{h}_1(x) = \frac{3}{2d^3}x^2\dot{x} = \frac{3}{2d^3}x^2\left(ax + \left(x^3 + b\right)u\right), \tag{2.62}$$

$$\dot{h}_2(x) = -\frac{3}{2d^3}x^2\dot{x} = -\frac{3}{2d^3}x^2\left(ax + \left(x^3 + b\right)u\right). \tag{2.63}$$

Notice that $\dot{h}_1(x)$ and $\dot{h}_2(x)$ depend on the control input u which cannot be known beforehand. Therefore, the assumption $\left|\dot{h}_{1,2}(z)\right| \le \phi_{1,2}$ in (Mozelli et al., 2009) is quite difficult to satisfy. The option is to give values to $\phi_{1,2}$ and verify a *posteriori* if these bound satisfy the assumption which is in detriment of the approach.♦

Following the ideas derived from the adoption of a FLF (2.19) and a non-PDC control law (2.51), condition $\dot{V}(x) < 0$ along the trajectories of the closed-loop TS model (2.53) is satisfied if:

$$P_h^{-1}\left(A_h + B_h F_h P_h^{-1}\right) + \left(A_h + B_h F_h P_h^{-1}\right)^T P_h^{-1} + P_{\dot{h}}^{-1} < 0 . \tag{2.64}$$

Multiplying left and right by P_h, the previous expression yields:

$$A_h P_h + B_h F_h + \left(A_h P_h + B_h F_h\right)^T + P_h P_{\dot{h}}^{-1} P_h < 0 , \tag{2.65}$$

from which the next rewriting can be done where $-P_{\dot{h}} = P_h P_{\dot{h}}^{-1} P_h$:

$$A_h P_h + B_h F_h + \left(A_h P_h + B_h F_h\right)^T - P_{\dot{h}} < 0 . \tag{2.66}$$

From (2.66), a fundamental property that links non-quadratic analysis with the existence of local solutions for stabilization proceeds as follows (Bernal et al., 2010).

Theorem 2.7. *(Local stabilizability)***:** The TS model (2.11) under the non-PDC control law (2.51) is locally asymptotically stable in a domain $\mathcal{D}$ including the origin, if there exist matrices $P_i = P_i^T > 0$ and F_i, $i \in \{1,2,\ldots,r\}$, such that $A_h P_h + P_h A_h^T + B_h F_h + F_h^T B_h^T < 0$ has a solution.

Now, recalling (2.39) and substituting in (2.66), the following inequality arise:

$$A_h P_h + B_h F_h + \left(A_h P_h + B_h F_h\right)^T - \sum_{k=1}^{p} \frac{\partial w_0^k}{\partial z_k} \dot{z}_k \left(P_{g_1(h,k)} - P_{g_2(h,k)}\right) < 0 . \tag{2.67}$$

Some works are derived from (2.67), all of them deal with $\dot{z}_k$ in different ways. In (Bernal et al., 2010), relationship $\dot{z}_k = \left(\frac{\partial z_k}{\partial x}\right)^T \dot{x} = \left(\frac{\partial z_k}{\partial x}\right)^T \left(A_h x + B_h u\right)$ is used as well as an extra assumption: a bound on the control law $\|u\| \le \mu$, which leads to the following theorem.

Theorem 2.8. The TS model (2.11) under the non-PDC control law (2.51) is locally asymptotically stable in the outermost Lyapunov level $\mathcal{D} = \left\{x : x^T P_h^{-1} x \le c\right\}$ contained both in the compact set $\mathcal{C}_x$ and $\bar{\mathcal{D}} = \bigcap_{v,k,v,l} \left\{ x : \left| \frac{\partial w_0^k}{\partial x_l} u_v \right| \le \mu \eta_{kl}, \left| \frac{\partial w_0^k}{\partial x_l} x_s \right| \le \lambda_{kls} \right\}$, if there exist matrices of proper size $P_j = P_j^T > x_0^2 I$, F_j, $j \in \{1,2,\ldots,r\}$, such that the following LMI conditions hold:

$$\begin{aligned}
&\begin{bmatrix} P_j & F_j^T \\ F_j & \mu^2 I_m \end{bmatrix} > 0, \ \forall j \in \{1,2,\cdots,r\} \\
&\Upsilon_{ii}^m < 0, \quad \forall i \in \{1,2,\cdots,r\}, \ \forall m \in \left\{1,2,\ldots,2^{pn_x(n_u+n_x)}\right\} \\
&\frac{2}{r-1}\Upsilon_{ii}^m + \Upsilon_{ij}^m + \Upsilon_{ji}^m < 0, \quad \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \ne j, \ \forall m \in \left\{1,2,\ldots,2^{pn_x(n_u+n_x)}\right\},
\end{aligned} \tag{2.68}$$

where

$\Upsilon_{ij}^m = A_i P_j + B_i F_j + P_j A_i^T + F_j^T B_i^T - \Theta_1 - \Theta_2$,

$\Theta_1 = \sum_{k=1}^{p} \sum_{l=1}^{n_x} \sum_{v=1}^{n_u} (-1)^{d_{kl(v+n_x)}^m} \eta_{kl} \mu (B_i)_{lv} \left(P_{g_1(j,k)} - P_{g_2(j,k)} \right)$

$\Theta_2 = \sum_{k=1}^{p} \sum_{l=1}^{n_x} \sum_{s=1}^{n_x} (-1)^{d_{kls}^m} \lambda_{kls} (A_i)_{ls} \left(P_{g_1(j,k)} - P_{g_2(j,k)} \right)$, $g_1(j,k)$ and $g_2(j,k)$ defined as in (2.39), $d_{kl(v+n_x)}^m$, d_{kls}^m defined from the binary representation of $m-1 = d_{pn_x(n_x+n_u)}^m + d_{pn_x(n_x+n_u-1)}^m \times 2 + \cdots + d_{111}^m \times 2^{pn_x(n_x+n_u)-1}$.

Results in Theorem 2.8 have been extended using a more complex non-PDC control law, instead of a PDC one, considering an extra term which depends on the time derivative of the MFs. This development leads to less conservative LMI conditions, details in (Guerra et al., 2012).

Although the approaches presented in (Bernal et al., 2010; Guerra et al., 2012), which consider a bound on the control law, overcome the problems presented in (Tanaka et al., 2003; Tanaka et al., 2007; Mozelli et al., 2009) about the necessity to know a priori the bounds of time-derivatives of MFs, they need to impose (even by LMIs) a bound on the control law, which constitutes a limitation.

The next result removes the assumption of bounding on the control law as $\|u\| \le \mu$ and changes the way to introduce the control law in the conditions. Recalling (2.66), which presents the conditions for non-quadratic stabilization:

$$A_h P_h + B_h F_h + \left(A_h P_h + B_h F_h \right)^T - P_{\dot{h}} < 0, \tag{2.69}$$

where $\dot{P}_h = \sum_{k=1}^{p} \frac{\partial w_0^k}{\partial z_k} \dot{z}_k \left(P_{g_1(h,k)} - P_{g_2(h,k)} \right)$.

As in the previous non-quadratic approaches the idea is to derive LMI conditions from (2.69). In (Guerra et al., 2011; Pan et al., 2012) the following bound is presented:

$$\left| \frac{\partial w_0^k}{\partial z_k} \dot{z}_k \right| \le \beta_k \,, \ \beta_k > 0 \,. \tag{2.70}$$

Because $\dot{z}_k(t) = \left(\frac{\partial z_k}{\partial x} \right)^T \dot{x}$ and substituting the closed-loop TS model (2.53), (2.70) is equivalent to:

$$\left| \frac{\partial w_0^k}{\partial z_k} \left(\frac{\partial z_k}{\partial x} \right)^T \left(A_h P_h + B_h F_h \right) P_h^{-1} x \right| \le \beta_k \,. \tag{2.71}$$

From (Pan et al., 2012), a state-space reduction, in the sense that not every state variable is concerned for each z_k, is defined as $\frac{\partial z_k}{\partial x} = T_k \frac{\partial z_k}{\partial \xi^k}$, which leads to the following bound:

$$\left| \frac{\partial w_0^k}{\partial z_k} \left(\frac{\partial z_k}{\partial \xi^k} \right)^T T_k^T \left(A_h P_h + B_h F_h \right) P_h^{-1} x \right| \le \beta_k \,. \tag{2.72}$$

Theorem 2.9. The TS model (2.11) under the non-PDC control law (2.51) is locally asymptotically stable in the outermost Lyapunov level $\mathcal{D} = \left\{ x : x^T P_h^{-1} x \le c \right\}$ contained both in the compact set $\mathcal{C}_x$ and $\bar{\mathcal{D}} = \left\{ x : x^T x \le \lambda_x^2 \right\}$, if given β_k there exist matrices of proper size $P_j = P_j^T > 0$, F_j, S_{ij}^k, $i, j \in \{1,2,\ldots,r\}$, $k \in \{1,2,\ldots,p\}$, such that the following LMI conditions hold:

$$\begin{aligned}
& I < \varphi_k P_j, \ \forall j \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\ldots,p\}, \\
& \Upsilon_{ii} < 0, \quad \forall i \in \{1,2,\cdots,r\}, \\
& \frac{2}{r-1} \Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \ne j, \\
& \Gamma_{ii}^k < 0, \quad \forall i \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\ldots,p\}, \\
& \frac{2}{r-1} \Gamma_{ii}^k + \Gamma_{ij}^k + \Gamma_{ji}^k < 0, \quad \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \ne j, \ \forall k \in \{1,2,\ldots,p\},
\end{aligned} \tag{2.73}$$

where $\Upsilon_{ij} = \begin{bmatrix} A_i P_j + B_i F_j + \left(A_i P_j + B_i F_j\right)^T + \frac{1}{2}\sum_{k=1}^{p} \beta_k^2 S_{ij}^k & (*) & (*) & (*) \\ P_{g_1(j,1)} - P_{g_2(j,1)} & -2S_{ij}^1 & (*) & (*) \\ \vdots & 0 & \ddots & (*) \\ P_{g_1(j,p)} - P_{g_2(j,p)} & 0 & 0 & -2S_{ij}^p \end{bmatrix}$, $\varphi_k = \frac{2\beta_k}{\lambda_k^2 + \lambda_x^2}$,

$\Gamma_{ij}^k = \begin{bmatrix} \varphi_k P_j & (*) \\ T_k^T \left(A_i P_j + B_i F_j\right) & I \end{bmatrix}$, $g_1(j,\bullet)$ and $g_2(j,\bullet)$ defined as in (2.39).

Based on FLF (2.19), the non-PDC control law (2.52), and the same bound stated in (2.72), the work developed in (Jaadari et al., 2012) gives LMI conditions via Finsler's lemma. Another point is that this approach removes the link between controller and Lyapunov function, which reduces conservativeness due to the slack variables introduced in the conditions. In the following theorem the main result of this work is presented.

Theorem 2.10. The TS model (2.11) under the non-PDC control law (2.54) is locally asymptotically stable in the outermost Lyapunov level $\mathcal{D} = \left\{x : x^T P_h^{-1} x \le c\right\}$ contained both in the compact set $\mathcal{C}_x$ and $\bar{\mathcal{D}} = \left\{x : x^T x \le \lambda_x^2\right\}$, if given β_k there exist a scalar $\varepsilon > 0$ and matrices of proper size $P_j = P_j^T > 0$, F_j, H_j, $S_j = S_j^T > 0$, $j \in \{1, 2, \ldots, r\}$, such that the following LMI conditions hold:

$$\begin{aligned} & I < \varphi_k S_j, \ \forall j \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\ldots,p\}, \\ & \varepsilon P_j - \sum_{j=1}^{r}\sum_{k=1}^{p} (-1)^{d_j^m} \beta_k \left(P_{g_1(j,k)} - P_{g_2(j,k)}\right) \le 0, \ \forall j,m \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\ldots,p\}, \\ & \Upsilon_{ii} < 0, \quad \forall i \in \{1,2,\cdots,r\}, \\ & \frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \ne j, \\ & \Gamma_{ii}^k < 0, \quad \forall i \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\ldots,p\}, \\ & \frac{2}{r-1}\Gamma_{ii}^k + \Gamma_{ij}^k + \Gamma_{ji}^k < 0, \quad \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \ne j, \ \forall k \in \{1,2,\ldots,p\}, \end{aligned} \tag{2.74}$$

where

$$\Upsilon_{ij} = \begin{bmatrix} A_iH_j + B_iF_j + \left(A_iH_j + B_iF_j\right)^T & (*) & (*) & (*) \\ H_j - P_j + \varepsilon\left(A_iH_j + B_iF_j\right) & -2\varepsilon P_j & (*) & (*) \\ \varepsilon F_j & 0 & -2\varepsilon I & (*) \\ \varepsilon H_j & 0 & 0 & -\varepsilon P_j \end{bmatrix}, \ \Gamma_{ij}^k = \begin{bmatrix} \varphi_k\left(H_j + H_j^T - S_j\right) & (*) \\ A_iH_j + B_iF_j & I \end{bmatrix},$$

$\varphi_k = \dfrac{2\beta_k}{\lambda_k^2 + \lambda_x^2}$, $g_1(j,k)$ and $g_2(j,k)$ defined as in (2.39), and d_j^m defined from the binary representation of $m-1 = d_r^m + d_{r-1}^m \times 2 + \cdots + d_1^m \times 2^{r-1}$.

Other works which achieve global conditions for TS model have been developed in (Rhee and Won, 2006; Mozelli et al., 2009). These approaches are based on line-integral Lyapunov function stated in (2.20). In (Rhee and Won, 2006) the following theorem is presented:

Theorem 2.11. The TS model (2.11) with the PDC control law (2.48) is asymptotically stable if there exist $\bar{P}$, $\mathfrak{D}_i$, F_i, and $X \geq 0$ such that the following conditions hold:

$$\begin{aligned} & P_i = \bar{P} + \mathfrak{D}_i > 0 \\ & G_{iii} + G_{iii}^T + (s-1)X < 0, \\ & G_{iij} + G_{iij}^T + \frac{1}{3}(s-3)X \leq 0, \ i \neq j, \\ & G_{ijk} + G_{ijk}^T - X \leq 0, \ i < j < k, \end{aligned} \tag{2.75}$$

where

$$G_{ijk} = \frac{1}{6}\left\{P_i\left(A_j + A_k + B_jF_k + B_kF_j\right) + P_j\left(A_i + A_k + B_iF_k + B_kF_i\right) + P_k\left(A_i + A_j + B_iF_j + B_jF_i\right)\right\},$$

$G_{iii} = P_i\left(A_i + B_iF_i\right)$, $G_{iij} = \frac{1}{3}\left\{P_i\left(A_i + A_j + B_iF_j + B_jF_i\right) + P_j\left(A_i + B_iF_i\right)\right\}$, $\bar{P}$ and $\mathfrak{D}_i$'s are defined as in (2.22), for $i, j, k = 1, 2, \ldots, r$.

Some difficulties arise for stabilization problem with the conditions presented in Theorem 2.11: 1) conditions are BMI, which are hard to solve since they may lead to local minima: a two-step algorithm is thus presented to solve it; 2) the TS model should be composed with a particular scheme on the MFs ($\omega_1^{\alpha_{i1}}(x_1)$, $\omega_2^{\alpha_{i2}}(x_2)$, …, $\omega_{n_x}^{\alpha_{in_x}}(x_{n_x})$) which restricts harshly the family of nonlinear systems under consideration; 3) a special structure is necessary to construct the Lyapunov matrices in order to satisfy path independent conditions.

In (Mozelli et al., 2009) an improvement for stabilization problem is presented where the conditions obtained are LMI instead of BMI as in (Rhee and Won, 2006). However, these new conditions are very restrictive because its special structure which enforces path-independency of the line integral.

2.6. Observer design for Takagi-Sugeno models

The problem of state estimation for dynamical systems is one of the main topics in control theory and has therefore been plentifully treated in the literature; its importance clearly arises from the fact that the control law often depends on state variables which may not be available due to the sensors high cost, inexistence, or impracticality. Observers are also useful for fault detection.

The following extension of the Luenberger observer presented in (Luenberger, 1971) is the most popular observer structure for nonlinear systems in a TS form:

$$\begin{aligned} \dot{\hat{x}} &= \sum_{i=1}^{r} h_i(\hat{z})\left(A_i\hat{x} + B_iu + K_i(y-\hat{y})\right) = A_{\hat{h}}\hat{x} + B_{\hat{h}}u + K_{\hat{h}}(y-\hat{y}) \\ \hat{y} &= \sum_{i=1}^{r} h_i(\hat{z})C_i\hat{x} = C_{\hat{h}}\hat{x}, \end{aligned} \tag{2.76}$$

with $\hat{x} \in \mathbb{R}^{n_x}$ as the estimated state, $\hat{y} \in \mathbb{R}^{n_y}$ as the estimated measured output, $\hat{z} \in \mathbb{R}^{n_p}$ as the estimated premise variable vector, and $K_i \in \mathbb{R}^{n_y \times n_x}$, $i \in \{1,2,\ldots,r\}$, being the gains of the observer to be designed.

In observer design, the estimated states converge asymptotically to the original states, this is $\hat{x} \to x$, as $t \to \infty$. In other words, the dynamics of the estimation error, defined as $e = x - \hat{x}$, must be stable and autonomous, which explains why design conditions are usually aimed to guarantee asymptotic stability of the estimation error.

Several works have been developed within the TS framework; these works can be separated in two classes: the first one considers that the premise vector $z = z(x)$ is built with measured variables (Tanaka et al., 1998; Patton et al., 1998; Teixeira et al., 2003; Akhenak et al., 2007; Z. Lendek et al., 2010); the second one assumes that the premise vector $\hat{z} = z(\hat{x})$ is also formed by unmeasured variables (Bergsten et al., 2001; Bergsten et al., 2002; Ichalal et al., 2007; Yoneyama, 2009; Z. Lendek et al., 2010; Ichalal et al., 2011; Ichalal et al., 2012). The main approaches about observer design for TS models are presented below.

2.6.1. Estimation of the state for Takagi-Sugeno models via measured variables

If the premise variables depend on measured variables z, then, the observer structure (2.76) yields:

$$
\begin{aligned}
\dot{\hat{x}} &= \sum_{i=1}^{r} h_i(z)\left(A_i\hat{x} + B_i u + P^{-1}K_i(y-\hat{y})\right) = A_h\hat{x} + B_h u + P^{-1}K_h(y-\hat{y}) \\
\hat{y} &= \sum_{i=1}^{r} h_i(z) C_i\hat{x} = C_h\hat{x}.
\end{aligned}
\tag{2.77}
$$

As for the control design P^{-1} is added, in order both to avoid any change of variable and to be suitable with various extensions. Therefore, the estimation error dynamics, $\dot{e} = \dot{x} - \dot{\hat{x}}$, is described as:

$$\dot{e} = \left(A_h - P^{-1}K_h C_h\right)e. \tag{2.78}$$

Consider the quadratic Lyapunov function candidate:

$$V(e) = e^T P e,\ P = P^T > 0. \tag{2.79}$$

The condition $\dot{V}(e) < 0$ is satisfied if:

$$PA_h - K_h C_h + A_h^T P - C_h^T K_h^T < 0. \tag{2.80}$$

The next theorem has been presented in (Tanaka et al., 1998) and gives LMI conditions to guarantee stability of the estimation error dynamics.

Theorem 2.12. The estimation error model (2.78) is asymptotically stable if there exist matrices $P = P^T > 0$ and K_j, $j \in \{1,\ldots,r\}$ of proper dimensions such that the following conditions hold:

$$
\begin{aligned}
&\Upsilon_{ii} < 0,\ \forall i \in \{1,2,\cdots,r\} \\
&\Upsilon_{ij} + \Upsilon_{ji} < 0,\ \forall (i,j) \in \{1,2,\cdots,r\}^2,\ i < j,
\end{aligned}
\tag{2.81}
$$

with $\Upsilon_{ij} = PA_i - K_j C_i + A_i^T P - C_i^T K_j^T$.

2.6.2. Estimation of the state for Takagi-Sugeno models via unmeasured variables

Consider that the premise variables depend on unmeasured variables $\hat{z}$ with common output matrices $C_i = C$, $i \in \{1, 2, \ldots, r\}$. Then, the observer structure (2.76) yields:

$$\begin{aligned} \dot{\hat{x}} &= \sum_{i=1}^{r} h_i(\hat{z})\left(A_i\hat{x} + B_i u + P^{-1}K_i(y-\hat{y})\right) = A_{\hat{h}}\hat{x} + B_{\hat{h}}u + P^{-1}K_{\hat{h}}(y-\hat{y}) \\ \hat{y} &= C\hat{x}. \end{aligned} \tag{2.82}$$

Therefore, the estimation error dynamics, $\dot{e} = \dot{x} - \dot{\hat{x}}$, is described as:

$$\dot{e} = A_h x - A_{\hat{h}}\hat{x} + B_h u - B_{\hat{h}}u - P^{-1}K_{\hat{h}}(Cx - C\hat{x}). \tag{2.83}$$

Provided $\hat{x} = x - e$, the previous expression can be rewritten as:

$$\dot{e} = \left(A_{\hat{h}} - P^{-1}K_{\hat{h}}C\right)e + \left(A_h - A_{\hat{h}}\right)x + \left(B_h - B_{\hat{h}}\right)u, \tag{2.84}$$

which can be expressed as:

$$\dot{e} = \sum_{i=1}^{r} h_i(\hat{z})\left(A_i - P^{-1}K_iC\right)e + \sum_{i=1}^{r}\left(h_i(z) - h_i(\hat{z})\right)\left(A_i x + B_i u\right). \tag{2.85}$$

It is clear that the membership function error term $h_i(z) - h_i(\hat{z})$ makes difficult to derive LMI conditions that guarantee the estimation error dynamics goes to zero.

One way to deal with this class of unmeasured variables is to consider the MF error $h_i(z) - h_i(\hat{z})$ and to use classical Lipschitz constants as in (Bergsten et al., 2001; Ichalal et al., 2007). The following result comes from (Bergsten et al., 2001).

Theorem 2.13. The estimation error model (2.85) is asymptotically stable if for a given scalar $\mu > 0$, there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, and K_i, $i \in \{1, \ldots, r\}$ of proper dimensions such that the following conditions hold:

$$\begin{aligned} &PA_i - K_iC + A_i^T P - C^T K_i^T + Q < 0, \quad \forall i \in \{1, 2, \cdots, r\} \\ &\begin{bmatrix} Q - \mu^2 I & P \\ P & I \end{bmatrix} > 0, \end{aligned} \tag{2.86}$$

assuming that:

$$\left\| \sum_{i=1}^{r} \left(h_i(x) - h_i(\hat{x}) \right) \left(A_i x + B_i u \right) \right\| \leq \mu \|e\|. \tag{2.87}$$

An alternative to deal with the problem has been presented in (Ichalal et al., 2007); in it, the following assumptions are made:

1. The matrices are calculated as: $A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i$, $\bar{A}_i = A_0 - A_i$.
2. The MFs are Lipschitz:

$$\left\| h_i(x) - h_i(\hat{x}) \right\| \leq N_i \|e\|. \tag{2.88}$$

$$\left\| h_i(x)x - h_i(\hat{x})\hat{x} \right\| \leq M_i \|e\|. \tag{2.89}$$

3. The input u is bounded: $\|u\| \leq \beta_1$, $\beta_1 > 0$.

Theorem 2.14. The estimation error model (2.85) is asymptotically stable if, for given scalars β_1, M_i, N_i, there exist scalars λ_1, λ_2, γ, and matrices of proper dimensions $P = P^T > 0$, $Q = Q^T > 0$, and K_i , $i \in \{1,\ldots,r\}$ such that the following conditions hold:

$$\begin{aligned} & PA_0 - K_i C + A_0^T P - C^T K_i^T + Q < 0, \ \forall i \in \{1,2,\cdots,r\} \\ & \begin{bmatrix} -Q + \lambda_1 M_i^2 I & P\bar{A}_i & PB_i & N_i \gamma I \\ (*) & -\lambda_1 I & 0 & 0 \\ (*) & 0 & -\lambda_2 I & 0 \\ (*) & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0, \\ & \gamma - \beta_1 \lambda_2 > 0. \end{aligned} \tag{2.90}$$

In (Ichalal et al., 2010), an alternative to the previous approach based on Lipschitz constants is presented as well as an extension to disturbance rejection. On the other hand, an approach based on $\mathcal{L}_2$ - gain has been developed in (Ichalal et al., 2008). This strategy gives better results than those based on Lipschitz constants. Nevertheless, only bounded error convergence is guaranteed.

Another way to face the problem of unmeasured variables is to use the Differential Mean Value Theorem (DMVT), by the mild assumption that the MFs $h_i(z)$ are class C^1. In this case, information about the MFs can be handled via the known bounds of its partial derivatives according to the state, i.e. $\frac{\partial h_i(z)}{\partial x}$; the approach in (Ichalal et al., 2011; Ichalal et al., 2012) follows this direction. Both the DMVT and the sector nonlinearity approach are used in order to get LMI conditions; details can be found in (Ichalal et al., 2011).

2.7. Asymptotically necessary and sufficient conditions in fuzzy control.

In many control problems for TS models, multiple (double, triple, and so on) nested convex sums appear. Therefore, in order to obtain LMI conditions from multiple-summation positivity (or negativity) problems, different relaxations have been developed which help dropping off the MFs from these expressions looking for the lowest possible conservativeness while preserving computational treatability. One of the first approaches for double summations has been proposed in (Wang et al., 1996) which has been enhanced in (Tuan et al., 2001) with an adequate compromise between quality and computational complexity. Another way to reduce the conservativeness of former approaches consisted on introducing slack variables in the conditions; see (Kim and Lee, 2000; Liu and Zhang, 2003; Teixeira et al., 2003) and (Fang et al., 2006) for double and triple sums, respectively. At last notice that the approaches presented in (Kim and Lee, 2000; Tuan et al., 2001; Liu and Zhang, 2003; Teixeira et al., 2003; Fang et al., 2006) are included as particular cases of results in (Sala and Ariño, 2007).

In the latter, based on Polya's theorem, a set of progressively less conservative sufficient conditions for proving positivity of fuzzy summations is provided. The main idea was to derive *asymptotically necessary and sufficient* (ANS) conditions for summations which are dependent on a *complexity parameter* d. It is important to point out that it asymptotically solves the sum relaxation problem, but it does **neither solve** the conservativeness coming from the use of a quadratic Lyapunov function **nor** from the type of TS model or control law employed. It is also shown in (Sala and Ariño, 2007) that, if parameter d increases to infinity, then conservativeness decreases to zero.

Some recalls about results presented in (Sala and Ariño, 2007) are presented below. Generally, for control design the following condition arises:

$$\Upsilon_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \Upsilon_{ij} > 0, \tag{2.91}$$

with $\Upsilon_{ij} = A_i P + B_i F_j + P A_i^T + F_j^T B_i^T$, for instance.

Then, conditions derived in (Sala and Ariño, 2007) came from the introduction of extra sums in (2.91)

$$\left[\Upsilon_{hh}\right]_d = \left(\sum_{i=1}^{r} h_i(z)\right)^d \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z) h_j(z) \Upsilon_{ij} > 0, \tag{2.92}$$

where $d \geq 0$ represents the *complexity parameter* for (2.92) and $\left[\Upsilon_{hh}\right]_d$ as the expansion of degree $d+2$. For instance, if $d = q-1$, then, (2.92) writes:

$$\sum_{i_1=1}^{r} \cdots \sum_{i_{q-1}=1}^{r} \sum_{i=1}^{r}\sum_{j=1}^{r} h_{i_1}(z) \cdots h_{i_{q-1}}(z) h_i(z) h_j(z) \Upsilon_{ij} > 0, \tag{2.93}$$

which represents $q+1$ fuzzy summations. Then, LMI conditions are obtained from (2.93) in the following way (relaxation lemma C.4):

$$\sum_{i_1 i_2 \cdots i_{q-1} ij \in \rho\left(i_1, i_2, \ldots, i_{q-1}, i, j\right)} \Upsilon_{ij} > 0, \quad \forall \left(i_1, i_2, \ldots, i_{q-1}, i, j\right) \in \{1, 2, \ldots, r\}^{q+1}, \tag{2.94}$$

with $\rho\left(i_1, i_2, \ldots, i_{q-1}, i, j\right)$ as the set of permutations with repeated elements of indexes $i_1, i_2, \ldots, i_{q-1}, i, j$.

Also, in (Sala and Ariño, 2007), an alternative introducing slack variables in the ANS conditions is proposed, such that for lower values of parameter d less conservative conditions arise respect to the conditions without the use of these decision variables. Nevertheless, the computational burden increases very quickly to the point where solvers fail, even for simple problems. On the other hand, despite of the fact that the approaches proposed in (Sala and Ariño, 2007) lead to necessary and sufficient conditions, a conservatism remains because of the selection of the Lyapunov function candidate or the particular control law scheme; there is therefore room for improvements.

On the other hand, in (Kruszewski et al., 2009) ANS conditions for summations are achieved considering a triangulation methodology to decide, in a finite number of steps, whether a given fuzzy control problem is strictly feasible or unfeasible; this approach has been improved in (Campos et al., 2012) due to the fact that the membership function shapes information are introduced in the methodology.

A completely different way to tackle with ANS was used in (Ding, 2010) for discrete-time TS models. The approach is concerned with increasing the complexity of the non-quadratic Lyapunov function via homogeneous polynomial parameter-dependent (HPPD) non-quadratic

Lyapunov function. It uses the property that HPDD non-quadratic Lyapunov function approximates asymptotically any smooth Lyapunov function. As said in the introduction these two last approaches have not being developed in this work.

2.8. Takagi-Sugeno Models in a Descriptor Form

The descriptor structure appeared in (Luenberger, 1977) with the main interest of describing nonlinear families of systems in a more natural way than the standard state-space one. In (Dai, 1989), some definitions are presented in order to determine the admissibility of the descriptor representation for linear case. Now, let us turn on the nonlinear case.

A nonlinear system affine-in-control in descriptor form is represented by:

$$\begin{aligned} E(x)\dot{x} &= A(x)x + B(x)u \\ y &= C(x)x + D(x)u, \end{aligned} \tag{2.95}$$

where $x \in \mathbb{R}^{n_x}$ represents the system state vector, $u \in \mathbb{R}^{n_u}$ the input vector, $y \in \mathbb{R}^{n_y}$ the measured output vector, and $E(x) \in \mathbb{R}^{n_x \times n_x}$, $A(x) \in \mathbb{R}^{n_x \times n_x}$, $B(x) \in \mathbb{R}^{n_x \times n_u}$, $C(x) \in \mathbb{R}^{n_y \times n_x}$, $D(x) \in \mathbb{R}^{n_y \times n_u}$ are the matrices of nonlinear functions. The matrix $E(x)$ might be singular in some cases. However, in this thesis only non-singular representation on $E(x)$ are addressed because numerous applications in mechanical systems can be treated with this assumption.

The sector nonlinearity approach has been applied in (Taniguchi et al., 1999) in order to get a TS representation of the descriptor model (2.95) which gives the next form:

$$\begin{aligned} E_v\dot{x} &= A_h x + B_h u \\ y &= C_h x + D_h u, \end{aligned} \tag{2.96}$$

where p_l and p_r are the number of nonlinear terms in left and right side, the sums $A_h = \sum_{i=1}^{r} h_i(z) A_i$, $B_h = \sum_{i=1}^{r} h_i(z) B_i$, $C_h = \sum_{i=1}^{r} h_i(z) C_i$, $D_h = \sum_{i=1}^{r} h_i(z) D_i$, and $E_v = \sum_{k=1}^{r_e} v_k(z) E_k$ depend on matrices of appropriate dimensions A_i, B_i, C_i, D_i, $i \in \{1, 2, \ldots, r\}$, and E_k, $k \in \{1, 2, \ldots, r_e\}$, $r = 2^{p_r}$ and $r_e = 2^{p_l}$ are the number of model rules in the left and right part, respectively. As in the ordinary TS model, the two sets of MFs $h_i(z) \geq 0$, $i \in \{1, \ldots, r\}$ and $v_k(z) \geq 0$, $k \in \{1, \ldots, r_e\}$ hold the convex sum property

$\sum_{i=1}^{r} h_i(z) = 1$ and $\sum_{k=1}^{r_e} v_k(z) = 1$ in a compact set of the state variables; they both depend on a premise vector $z \in \mathbb{R}^p$ which depends on the state x.

The descriptor structure has sometimes real advantages in reducing the number of LMI constraints in contrast with standard TS modeling, thus alleviating the computational burden, as the following example illustrates.

Example 2.4. Consider a descriptor nonlinear model with the following matrices:

$$E(x)\dot{x} = A(x)x, \tag{2.97}$$

where $E(x) = \begin{bmatrix} \dfrac{1}{1+x_1^2} & 1 \\ -1 & 1 \end{bmatrix}$ and $A(x) = \begin{bmatrix} -1 & 0 \\ x_2^2 & -3+\sin(x_1) \end{bmatrix}$.

Equation (2.97) can be rewritten as a standard TS model using $E(x)^{-1}$ as follows:

$$\dot{x}(t) = E(x)^{-1} A(x) x(t), \tag{2.98}$$

with $E(x)^{-1} = \dfrac{1}{2+x_1^2}\begin{bmatrix} 1+x_1^2 & -1-x_1^2 \\ 1+x_1^2 & 1 \end{bmatrix}$.

The descriptor representation (2.97) gives $r = 2^2 = 4$ because the nonlinear terms x_2^2 and $\sin(x_1)$ in $A(x)$, and $r_e = 2^1 = 2$ due to $\dfrac{1}{1+x_1^2}$ in $E(x)$. On the other hand, in the standard representation (2.98) the nonlinear terms after multiplication in the right-hand side lead to $r = 2^4 = 16$. When stability analysis under quadratic framework is addressed, the number of LMI conditions to be satisfied depends on the number of rules for each case; for this example, it is necessary to satisfy 33 and 257 LMI conditions for descriptor and standard representations, respectively, which shows an important reduction in computational terms if a TS model in the descriptor form is used.♦

Some results concerned with stability and controller/observer design (Taniguchi et al., 1999; Taniguchi et al., 2000; Guerra et al., 2004; Guerra et al., 2007) for TS models in a descriptor form have been addressed. The main results in this direction are presented below.

2.8.1. Stability of Descriptor Takagi-Sugeno Models

Consider the descriptor TS model in (2.96) with $u=0$, which can be rewritten through the extended vector $\bar{x}=\begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$ as:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}_{hv}\bar{x}(t)+\bar{B}_h u(t) \\ y(t) &= \bar{C}_h \bar{x}(t)+\bar{D}_h u(t), \end{aligned} \tag{2.99}$$

with $\bar{E}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv}=\begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h=\begin{bmatrix} 0 \\ B_h \end{bmatrix}$, $\bar{C}_h=\begin{bmatrix} C_h & 0 \end{bmatrix}$, and $\bar{D}_h=\begin{bmatrix} 0 \\ D_h \end{bmatrix}$.

Consider a quadratic Lyapunov function candidate:

$$V(\bar{x})=\bar{x}^T\bar{E}^T\bar{P}\bar{x}, \ \bar{E}^T\bar{P}>0, \tag{2.100}$$

where structure of $\bar{P}$ is selected such that $\bar{E}^T\bar{P}=\bar{P}^T\bar{E}$ is satisfied.

Then, the time-derivative of $V(\bar{x})$ is:

$$\begin{aligned} \dot{V}(\bar{x}) &= \dot{\bar{x}}^T\bar{E}^T\bar{P}\bar{x}+\bar{x}^T\bar{E}^T\bar{P}\dot{\bar{x}}+\bar{x}^T\bar{E}^T\dot{\bar{P}}\bar{x} \\ &= \dot{\bar{x}}^T\bar{E}^T\bar{P}\bar{x}+\bar{x}^T\bar{P}^T\bar{E}\dot{\bar{x}}+\bar{x}^T\bar{E}^T\dot{\bar{P}}\bar{x}. \end{aligned} \tag{2.101}$$

In order to satisfy $\bar{E}^T\bar{P}=\bar{P}^T\bar{E}$ some structures on $\bar{P}$ have been defined in the literature: 1) with a constant matrix $\bar{P}=\begin{bmatrix} P^1 & 0 \\ P^3 & P^1 \end{bmatrix}$ where $P^1=\left(P^1\right)^T>0$ (Taniguchi et al., 2000); 2) with a constant matrix $\bar{P}=\begin{bmatrix} P^1 & 0 \\ P^3 & P^4 \end{bmatrix}$ where $P^1=\left(P^1\right)^T>0$, P^3 and P^4 as free matrices (Guerra et al., 2004); 3) with a convex matrix $\bar{P}=\begin{bmatrix} P^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix}$ where $P^1=\left(P^1\right)^T>0$, $P_h^3=\sum_{j=1}^{r}h_jP_j^3$ and $P_h^4=\sum_{j=1}^{r}h_jP_j^4$ convex matrices (Guerra et al., 2007). Case 3) gives a better set of solutions than the other cases. Note that whatever is the structure selected, $P^1=\left(P^1\right)^T>0$ has been always chosen as a constant matrix in order to avoid the time derivatives of the MFs; this guarantees $\bar{E}^T\dot{\bar{P}}=0$ and $\bar{E}^T\bar{P}=\bar{P}^T\bar{E}>0$.

Selecting case 3) for a descriptor TS model (2.99) with $u=0$, the time-derivative $\dot{V}(\bar{x})<0$ is satisfied if:

$$\bar{A}_{hv}^T \bar{P} + \bar{P}^T \bar{A}_{hv} < 0, \tag{2.102}$$

leading to the next theorem.

Theorem 2.15. The descriptor TS model (2.99) with $u = 0$ is asymptotically stable if there exists a common matrix $P_1 = P_1^T > 0$, and matrices P_j^3 and P_j^4, such that:

$$\begin{aligned} &\Upsilon_{ii}^k < 0, \ \forall i \in \{1,2,\cdots,r\}, \ \forall k \in \{1,2,\cdots,r_e\} \\ &\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \ \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \neq j, \ \forall k \in \{1,2,\cdots,r_e\}, \end{aligned} \tag{2.103}$$

with $\begin{bmatrix} A_i^T P_j^3 + \left(P_j^3\right)^T A_i & (*) \\ P^1 - E_k P_j^3 + \left(P_j^4\right)^T A_i & -E_k P_j^4 - \left(E_k P_j^4\right)^T \end{bmatrix} < 0$.

2.8.2. Stabilization of Descriptor Takagi-Sugeno Models

Consider the following modified PDC control law:

$$u(t) = F_{hv}\left(P^1\right)^{-1} x(t) = \begin{bmatrix} F_{hv}\left(P^1\right)^{-1} & 0 \end{bmatrix} \bar{x}(t) = \bar{F}_{hv}\bar{x}(t), \tag{2.104}$$

with $F_{hv} = \sum_{j=1}^{r}\sum_{k=1}^{r_e} h_j v_k F_{jk}$ formed by the gains F_{jk}, $j \in \{1,\ldots,r\}$, $k \in \{1,\ldots,r_e\}$.

When the PDC control law (2.104) is substituted in the state equation (2.99), the following closed-loop descriptor TS model is obtained:

$$\bar{E}\dot{\bar{x}} = \left(\bar{A}_{hv} + \bar{B}_h \bar{F}_{hv}\right)\bar{x}. \tag{2.105}$$

In order to derive LMI conditions, a first result presented in (Taniguchi et al., 2000) with the following quadratic Lyapunov function was developed:

$$V(\bar{x}) = \bar{x}^T \bar{E}^T \bar{P}^{-1}\bar{x}, \ \bar{E}^T \bar{P}^{-1} = \bar{P}^{-T}\bar{E} > 0, \tag{2.106}$$

where $\bar{P} = \begin{bmatrix} P^1 & 0 \\ P^3 & P^1 \end{bmatrix}$, $P^1 = \left(P^1\right)^T > 0$.

Following the same path of the stability case, the next conditions guarantee $\dot{V}(\bar{x}) < 0$:

$$\bar{P}^{-T}\left(\bar{A}_{hv} + \bar{B}_h \bar{F}_{hv}\right) + \left(\bar{A}_{hv} + \bar{B}_h \bar{F}_{hv}\right)^T \bar{P}^{-1} < 0. \tag{2.107}$$

Multiplying the previous expression by $\bar{P}^T$ on the left-hand side and by its transpose $\bar{P}$ on the right-hand side (the matrix is not symmetric), gives

$$\bar{A}_{hv}\bar{P} + \bar{B}_h\bar{F}_{hv}\bar{P} + \bar{P}^T\bar{A}_{hv}^T + \bar{P}^T\bar{F}_{hv}^T\bar{B}_h^T < 0\,. \tag{2.108}$$

Recalling the definitions of $\bar{A}_{hv}$, $\bar{B}_h$, $\bar{F}_{hv}$ and $\bar{P}$, the previous conditions are equivalent to:

$$\begin{bmatrix} P^3 + \left(P^3\right)^T & (*) \\ A_h P^1 + B_h F_{hv} - E_v P^3 + \left(P^1\right)^T & -E_v P^1 - \left(E_v P^1\right)^T \end{bmatrix} < 0\,. \tag{2.109}$$

A more general structure of $\bar{P}$ has been proposed in (Guerra et al., 2007). If $\bar{P} = \begin{bmatrix} P^1 & 0 \\ P_{hh}^3 & P_{hh}^4 \end{bmatrix}$ with $P_{hh}^3 = \sum_{i=1}^r \sum_{j=1}^r h_i h_j P_{ij}^3$, $P_{hh}^4 = \sum_{i=1}^r \sum_{j=1}^r h_i h_j P_{ij}^4$ as a regular matrix, and applying the same steps for conditions (2.109), the following are obtained:

$$\begin{bmatrix} P_{hh}^3 + \left(P_{hh}^3\right)^T & (*) \\ A_h P^1 + B_h F_{hv} - E_v P_{hh}^3 + \left(P_{hh}^4\right)^T & -E_v P_{hh}^4 - \left(E_v P_{hh}^4\right)^T \end{bmatrix} < 0\,, \tag{2.110}$$

which after removal of the MFs through relaxations lead to the following theorem.

Theorem 2.16. The descriptor TS model (2.99) is asymptotically stable if there exist matrices $P_1 = P_1^T > 0$, P_{ij}^3, and P_{ij}^4 such that:

$$\begin{gathered} \Upsilon_{ii}^k < 0,\ \forall i \in \{1,2,\cdots,r\},\ \forall k \in \{1,2,\cdots,r_e\} \\ \frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0,\ \forall (i,j) \in \{1,2,\cdots,r\}^2,\ i \neq j,\ \forall k \in \{1,2,\cdots,r_e\}, \end{gathered} \tag{2.111}$$

with $\begin{bmatrix} P_{ij}^3 + \left(P_{ij}^3\right)^T & (*) \\ A_i P^1 + B_i F_{jk} - E_k P_{ij}^3 + \left(P_{ij}^4\right)^T & -E_k P_{ij}^4 - \left(E_k P_{ij}^4\right)^T \end{bmatrix}$.

Remark 2.12. Regularity of P_{hh}^4 derives directly from block entry $(2,2)$ of (2.111). If conditions in (2.111) hold, then (2.110) holds, which implies:

$$-E_v P_{hh}^4 - \left(E_v P_{hh}^4\right)^T < 0\,. \tag{2.112}$$

Suppose that P_{hh}^4 is singular; therefore, it exists $x \neq 0$ such that $P_{hh}^4 x = 0$. Consider such $x \neq 0$ for (2.112); then:

$$x^T\left(-E_v P_{hh}^4 - \left(E_v P_{hh}^4\right)^T\right)x = 0\,, \tag{2.113}$$

which contradicts (2.112) and ends the proof.

2.9. Concluding Remark

This chapter has provided the main results for a class of nonlinear systems described in a TS form. Several results on stability analysis and controller design under a quadratic and non-quadratic LMI framework have been presented highlighting the principal contributions and drawbacks of these approaches. Also, the state estimation problem for dynamical systems has been treated both for measured and unmeasured premise variables. In addition, descriptor TS model scheme and various proposals about it have been summarized. Some examples were given to clarify the concepts and approaches.

The following problems will be addressed in the next chapters in order to provide some proposal of solution to tackle them:

- Despite of the fact that *asymptotically necessary and sufficient* (ANS) conditions are provided in the literature, the high demand of computational resources as well as the conservatism associated to the choice of the Lyapunov function candidate or the particular control law scheme are still open problems.
- Even with the use of non-quadratic Lyapunov functions to reduce conservativeness of the sufficient conditions because the quadratic scheme, in the continuous-time case arise the necessity of handling the time-derivatives of MFs which difficult to find global conditions to the problem on controller design.
- Observer design for TS models under unmeasured premise variables which is not easy to cast as a convex problem.

CHAPTER 3. Controller design for Takagi-Sugeno Models

3.1. Introduction

This chapter presents some contributions on state feedback controller design for continuous-time nonlinear systems. The methodologies are based on exact TS representations of the nonlinear setups under consideration; both standard as well as descriptor forms are addressed.

The first part is about standard TS models. The controller design schemes are based on: 1) a quadratic Lyapunov function (QLF) (Tanaka and Wang, 2001); 2) a fuzzy Lyapunov function (FLF) (Tanaka et al., 2003); 3) a line-integral Lyapunov function (LILF) (Rhee and Won, 2006); 4) a novel non-quadratic Lyapunov functional (NQLF). Schemes 1) and 2) incorporate a sum relaxation scheme based on multiple convex sums as the one in (Sala and Ariño, 2007); in this case the improvements come from matrix transformations such as those in (Shaked, 2001), (de Oliveira and Skelton, 2001), and (Peaucelle et al., 2000). Extensions to $H\infty$ performance design are made.

The second part concerns TS descriptor models. Two strategies are proposed: 1) within the quadratic framework, conditions based on a general control law and the matrix transformation in (Peaucelle et al., 2000); an extension to $H\infty$ disturbance rejection is presented; 2) an extension to the non-quadratic approach in (Rhee and Won, 2006) for second-order systems, which uses a line-integral Lyapunov function (LILF), a non-PDC control law, and the Finsler's Lemma; this strategy offers parameter-dependent LMI conditions instead of BMI constraints. Improvements are shown via illustrative examples along the chapter.

3.2. State feedback controller design for standard TS models

This section presents some schemes to tackle the stabilization problem as well as $H\infty$ disturbance rejection of continuous-time standard TS models. The proposals developed below give more relaxed conditions than former results.

3.2.1. Problem statement

Consider the following continuous-time T-S model with disturbances acting on the state and output equations:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} h_i(z)\left(A_i x + B_i u + D_i w\right) = A_h x + B_h u + D_h w \\ y &= \sum_{i=1}^{r} h_i(z)\left(C_i x + J_i u + G_i w\right) = C_h x + J_h u + G_h w, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^{n_x}$ represents the system state vector, $u \in \mathbb{R}^{n_u}$ the input vector, $y \in \mathbb{R}^{n_y}$ the measured output vector, $w \in \mathbb{R}^{n_w}$ the vector of external disturbances; $h_i(\cdot)$, $i \in \{1, 2, \ldots, r\}$ the membership functions which depend on the vector of premise variables grouped in $z \in \mathbb{R}^p$; and matrices $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n_x}$, $D_i \in \mathbb{R}^{n_x \times n_w}$, $J_i \in \mathbb{R}^{n_y \times n_u}$, and $G_i \in \mathbb{R}^{n_y \times n_w}$ result from the TS modeling of an associated nonlinear system (for instance, via the sector nonlinearity approach).

Control laws with the following general form will be adopted:

$$u = \mathcal{F}(z)\mathcal{H}^{-1}(z)x, \tag{3.2}$$

where $\mathcal{H}(z) \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{F}(z) \in \mathbb{R}^{n_x \times n_y}$ are matrix functions of the premise vector z to be designed in the sequel. Some of the results obtained are based on a procedure that "decouples" the control matrices from the Lyapunov function.

Then, the following closed-loop model arises:

$$\begin{aligned} \dot{x} &= \left(A_h + B_h \mathcal{F}\mathcal{H}^{-1}\right)x + D_h w \\ y &= \left(C_h + J_h \mathcal{F}\mathcal{H}^{-1}\right)x + G_h w. \end{aligned} \tag{3.3}$$

Remark 3.1: In the following sections, matrix functions $\mathcal{H}(z)$ and $\mathcal{F}(z)$ are going to be chosen according to the problem under consideration, attending to the following criteria: (a) do they lead to LMI conditions?, (b) do they relax (i.e., contain) previous results?, (c) do they provide any insight towards the goal of closing the gap between "classical nonlinear control" and TS-based nonlinear control?

3.2.2. Stabilization via quadratic Lyapunov function

Three results which reduce conservativeness without leaving the quadratic framework are hereby presented: a first one based on a Tustin-like transformation (Shaked, 2001), another one exploiting the Finsler's Lemma (de Oliveira and Skelton, 2001), and a last one using the matrix transformation in (Peaucelle et al., 2000). All of them share the characteristics of using slack variables to relax existing conditions as well as being compatible with a control law whose complexity can be increased (up to computational limitations) to obtain progressively less conservative results (Márquez et al., 2013a).

Why is it relevant to introduce control laws whose complexity may lead to less conservative conditions if there are already *asymptotic necessary and sufficient* (ANS) conditions for quadratic PDC-based controller design? The reason lies on the fact that ANS conditions have been obtained **only** for *convex summations* (Sala and Ariño, 2007) whose computational burden reaches very quickly a prohibitive size for current solvers; thus, approaches preserving asymptotic characteristics while reaching solutions where ANS conditions cannot, are worth exploring. The following example illustrates the limitations of the ANS methods.

Example 3.1: This example is constructed as follows (Delmotte et al., 2007): consider a TS representation with 2 models

$$A_1 = \begin{bmatrix} 0.5 & 0 \\ 1 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & -1 \\ -1 & 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \tag{3.4}$$

that is proved to be stabilizable via an ordinary PDC control law and a quadratic Lyapunov function. Complexity in the representation can be introduced artificially by adding models inside the original polytope. The matrices thus obtained are equally spaced, i.e.: $\left(A_{\delta_k}, B_{\delta_k}\right)$, $\delta_k = \dfrac{k}{r-1}$ with $k \in \{1, 2, \ldots, r-2\}$ corresponds to:

$$A_{1+\delta_k} = \left(1-\delta_k\right) A_1 + \delta_k A_2,\ B_{1+\delta_k} = \left(1-\delta_k\right) B_1 + \delta_k B_2. \tag{3.5}$$

Thus, the quadratic stabilizability via a PDC control law is guaranteed independently of the number of models r. Stabilization conditions in (Sala and Ariño, 2007) use Polya's property

(Scherer, 2006) introducing extra sums in the initial problem $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Upsilon_{ij} < 0$ with $\Upsilon_{ij} = A_i P + B_i F_j + (*)$, i.e.:

$$\left(\sum_{i=1}^{r} h_i(z)\right)^d \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Upsilon_{ij} < 0, \tag{3.6}$$

where d represents the *complexity parameter* for (3.6). Note that if there exists a solution to the initial problem there *must exist* a sufficiently large value of d such that the problem (3.6) is feasible. Theorem 5 of (Sala and Ariño, 2007) also adds some extra variables relaxing the conditions for a fixed value of d. Despite its simplicity, conditions in Theorem 5 of (Sala and Ariño, 2007) with $r = 10$ and $d = 2$, lead LMI solvers to failure. In this case, the number of LMI conditions and scalar decision variables are 41123 and 772, respectively. This example shows that, sometimes, very simple problems cannot be solved even if ANS conditions are available. ♦

The approaches to be developed will improve over existing ANS conditions since they reduce the computational burden (to help numerical solvers) and include Polya's theorem conditions (3.6) as a particular case (to maintain ANS property) due to the matrix transformations.

Consider the following quadratic Lyapunov function (QLF) candidate with $P = P^T > 0$:

$$V(x) = x^T P^{-1} x. \tag{3.7}$$

Condition $\dot{V}(x) < 0$ is satisfied if

$$x^T P^{-1}\dot{x} + \dot{x}^T P^{-1} x < 0. \tag{3.8}$$

The notation for "q" multiple nested convex sums, given in Table 2.1, will be used in the sequel; i.e.: $\Upsilon_{\bar{h}} = \Upsilon_{\underbrace{hh\cdots h}_{q}} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}(z)h_{i_2}(z)\cdots h_{i_q}(z)\Upsilon_{i_1 i_2\cdots i_q}$.

Tustin-like transformation

Using (3.3) with $w = 0$, condition in (3.8) is equivalent to:

$$P^{-1}\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right) + \left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T P^{-1} < 0. \tag{3.9}$$

The following development follows the same line of (Shaked, 2001): pre- and post-multiplying the previous expression by P and considering a small enough $\varepsilon > 0$, the following condition is also equivalent:

$$\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)P + P\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T + \varepsilon\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)P^{-1}\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T < 0, \quad (3.10)$$

from which the next rewriting can be done multiplying by ε and adding $P - P$:

$$\left(I + \varepsilon\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)\right)P\left(I + \varepsilon\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)\right)^T - P < 0, \quad (3.11)$$

which by Schur complement will be equivalent to:

$$\begin{bmatrix} P & I + \varepsilon\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right) \\ I + \varepsilon\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T & P^{-1} \end{bmatrix} > 0. \quad (3.12)$$

The previous expression can be pre-multiplied by $\begin{bmatrix} I & 0 \\ 0 & \mathcal{H}^T \end{bmatrix}$ and post-multiplied by $\begin{bmatrix} I & 0 \\ 0 & \mathcal{H} \end{bmatrix}$ to produce the equivalent condition:

$$\begin{bmatrix} P & \mathcal{H} + \varepsilon\left(A_h\mathcal{H} + B_h\mathcal{F}\right) \\ \mathcal{H}^T + \varepsilon\left(A_h\mathcal{H} + B_h\mathcal{F}\right)^T & \mathcal{H}^T P^{-1}\mathcal{H} \end{bmatrix} > 0. \quad (3.13)$$

Selecting $\mathcal{H} = H_{\bar{h}}$ and $\mathcal{F} = F_{\bar{h}}$, the following theorem is stated.

Theorem 3.1. The TS model (3.1) with $w = 0$ under control law (3.2) with $\mathcal{H} = H_{\bar{h}}$ and $\mathcal{F} = F_{\bar{h}}$ is globally asymptotically stable if $\exists \varepsilon > 0$ and matrices $P = P^T > 0$, $H_{i_1 i_2 \cdots i_q}$, and $F_{i_1 i_2 \cdots i_q}$, $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, r\}$ of proper dimensions such that the following conditions hold:

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \rho\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall\left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1, 2, \ldots, r\}^{q+1}, \quad (3.14)$$

with $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} -P & H_{i_1 i_2 \cdots i_q} + \varepsilon\left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q}\right) \\ (*) & P - H_{i_1 i_2 \cdots i_q} - H_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$ and $\rho\left(i_0, i_1, i_2, \ldots, i_q\right)$ as the set of permutations with repeated elements of indexes $i_0, i_1, i_2, \ldots, i_q$.

Proof: Using the property B.7 with $\mathcal{Q}=\mathcal{H}=H_{\bar{h}}$ and $\mathcal{P}=P$, it is clear that $H_{\bar{h}}P^{-1}H_{\bar{h}}^{T} \geq H_{\bar{h}}+H_{\bar{h}}^{T}-P$, which allows guaranteeing (3.13) if the following holds:

$$\begin{bmatrix} -P & H_{\bar{h}}+\varepsilon\left(A_h H_{\bar{h}}+B_h F_{\bar{h}}\right) \\ (*) & P-H_{\bar{h}}-H_{\bar{h}}^{T} \end{bmatrix}<0. \tag{3.15}$$

Applying the relaxation lemma C.4 to (3.15) leads to conditions (3.14), which concludes the proof. ◈

Finsler's Lemma

A different way is to consider the Finsler's Lemma. For this approach, condition in (3.8) can be rearranged as

$$\dot{V}=\begin{bmatrix} x \\ \dot{x} \end{bmatrix}^{T}\begin{bmatrix} 0 & P^{-1} \\ P^{-1} & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}<0, \tag{3.16}$$

altogether with the following equality restriction

$$\begin{bmatrix} A_h+B_h\mathcal{F}\mathcal{H}^{-1} & -I \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}=0, \tag{3.17}$$

which comes from (3.3) with $w=0$. If inequality (3.16) under equality constraint (3.17) holds, it is equivalent, by Finsler's Lemma, to the existence of matrices $\mathcal{V}$ and $\mathcal{W}$, such that

$$\begin{bmatrix} 0 & P^{-1} \\ P^{-1} & 0 \end{bmatrix}+\begin{bmatrix} \mathcal{V} \\ \mathcal{W} \end{bmatrix}\begin{bmatrix} A_h+B_h\mathcal{F}\mathcal{H}^{-1} & -I \end{bmatrix}+(*)<0. \tag{3.18}$$

Pre- and post-multiplying by $\begin{bmatrix} \mathcal{H}^{T} & 0 \\ 0 & P \end{bmatrix}$ and $\begin{bmatrix} \mathcal{H} & 0 \\ 0 & P \end{bmatrix}$ allows the following to be obtained

$$\begin{bmatrix} 0 & \mathcal{H}^{T} \\ \mathcal{H} & 0 \end{bmatrix}+\begin{bmatrix} \mathcal{H}^{T}\mathcal{V} \\ P\mathcal{W} \end{bmatrix}\begin{bmatrix} A_h\mathcal{H}+B_h\mathcal{F} & -P \end{bmatrix}+(*)<0. \tag{3.19}$$

In order to get LMI conditions and recover the "classical" quadratic case, let $\mathcal{V}=\mathcal{H}^{-T}$ and $\mathcal{W}=\varepsilon P^{-1}$ with $\varepsilon>0$; then (3.19) holds if

$$\begin{bmatrix} A_h\mathcal{H}+B_h\mathcal{F}+\left(A_h\mathcal{H}+B_h\mathcal{F}\right)^{T} & (*) \\ \mathcal{H}-P+\varepsilon\left(A_h\mathcal{H}+B_h\mathcal{F}\right) & -2\varepsilon P \end{bmatrix}<0 \tag{3.20}$$

and choosing $\mathcal{H}=H_{\bar{h}}$ and $\mathcal{F}=F_{\bar{h}}$, the following theorem arises.

Theorem 3.2. The TS model (3.1) with $w=0$ under control law (3.2) with $\mathcal{H}=H_{\bar{h}}$ and $\mathcal{F}=F_{\bar{h}}$ is globally asymptotically stable if $\exists\varepsilon>0$ and matrices of proper dimensions $P=P^T>0$, $H_{i_1i_2\cdots i_q}$, and $F_{i_1i_2\cdots i_q}$, $i_1,i_2,\ldots,i_q\in\{1,2,\ldots,r\}$, such that the next conditions hold:

$$\sum_{i_0i_1i_2\cdots i_q\in\wp\left(i_0i_1i_2\cdots i_q\right)}\Upsilon_{i_0i_1i_2\cdots i_q}<0,\quad\forall\left(i_0,i_1,i_2,\ldots,i_q\right)\in\{1,2,\ldots,r\}^{q+1},\tag{3.21}$$

with $\Upsilon_{i_0i_1i_2\cdots i_q}=\begin{bmatrix}A_{i_0}H_{i_1i_2\cdots i_q}+B_{i_0}F_{i_1i_2\cdots i_q}+\left(A_{i_0}H_{i_1i_2\cdots i_q}+B_{i_0}F_{i_1i_2\cdots i_q}\right)^T & (*)\\ H_{i_1i_2\cdots i_q}-P+\varepsilon\left(A_{i_0}H_{i_1i_2\cdots i_q}+B_{i_0}F_{i_1i_2\cdots i_q}\right) & -2\varepsilon P\end{bmatrix}$.

Proof: After substitution of $\mathcal{H}=H_{\bar{h}}$ and $\mathcal{F}=F_{\bar{h}}$ in (3.20), it yields

$$\begin{bmatrix}A_hH_{\bar{h}}+B_hF_{\bar{h}}+\left(A_hH_{\bar{h}}+B_hF_{\bar{h}}\right)^T & (*)\\ H_{\bar{h}}-P+\varepsilon\left(A_hH_{\bar{h}}+B_hF_{\bar{h}}\right) & -2\varepsilon P\end{bmatrix}<0.\tag{3.22}$$

Applying lemma C.4 to (3.22) gives the conditions presented in (3.21); thus producing the desired result. ◈

Matrix transformation (Peaucelle et al., 2000)

Another possibility is to follow a similar path as in (Peaucelle et al., 2000). In this case, condition in (3.8) is equivalent to:

$$P^{-1}\left(A_h+B_h\mathcal{F}\mathcal{H}^{-1}\right)+\left(A_h+B_h\mathcal{F}\mathcal{H}^{-1}\right)^T P^{-1}<0.\tag{3.23}$$

After congruence property with P, the last inequality writes

$$A_hP+B_h\mathcal{F}\mathcal{H}^{-1}P+\left(A_hP+B_h\mathcal{F}\mathcal{H}^{-1}P\right)^T<0,\tag{3.24}$$

which leads to the next theorem.

Theorem 3.3. The TS model (3.1) with $w=0$ under control law (3.2) with $\mathcal{H}=P$ and $\mathcal{F}=F_{\bar{h}}$ is globally asymptotically stable if there exist matrices of proper dimensions $P=P^T>0$, $R_{i_1i_2\cdots i_q}$, $H_{i_1i_2\cdots i_q}$, and $K_{i_1i_2\cdots i_q}$, $i_1,i_2,\ldots,i_q\in\{1,2,\ldots,r\}$ such that the following conditions hold:

$$\sum_{i_0i_1i_2\cdots i_q\in\wp\left(i_0i_1i_2\cdots i_q\right)}\Upsilon_{i_0i_1i_2\cdots i_q}<0,\quad\forall\left(i_0,i_1,i_2,\ldots,i_q\right)\in\{1,2,\ldots,r\}^{q+1},\tag{3.25}$$

with $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} + \left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} \right)^T & (*) \\ P - H_{i_1 i_2 \cdots i_q} + R^T_{i_1 i_2 \cdots i_q} A^T_{i_0} & -R_{i_1 i_2 \cdots i_q} - R^T_{i_1 i_2 \cdots i_q} \end{bmatrix}$.

Proof: Assuming $\mathcal{H} = P$, $\mathcal{F} = F_{\bar{h}}$, and applying property B.3 with $\mathcal{A} = A_h^T$, $\mathcal{L}^T = H_{\bar{h}}$, $\mathcal{R} = R_{\bar{h}}$, $\mathcal{P} = P$, $\mathcal{Q} = B_h F_{\bar{h}} + F_{\bar{h}}^T B_h^T$ to (3.24), it writes:

$$\begin{bmatrix} A_h H_{\bar{h}} + B_h F_{\bar{h}} + \left(A_h H_{\bar{h}} + B_h F_{\bar{h}} \right)^T & (*) \\ P - H_{\bar{h}} + R_{\bar{h}}^T A_h^T & -R_{\bar{h}} - R_{\bar{h}}^T \end{bmatrix} < 0. \tag{3.26}$$

When relaxation lemma C.4 is applied to the previous expression, conditions in (3.25) are obtained, which ends the proof. ◈

Remark 3.2: Conditions in (3.14) and (3.21) are LMI once the parameter $\varepsilon > 0$ is chosen. This parameter is employed in several works concerning linear parameter varying (LPV) systems (de Oliveira and Skelton, 2001; Shaked, 2001; Oliveira et al., 2011; Jaadari et al., 2012): they are normally prefixed values belonging to a logarithmically spaced family of values, such as: $\varepsilon \in \left\{ 10^{-6}, 10^{-5}, \ldots, 10^{6} \right\}$, which avoids performing an exhaustive linear search. This set of values for ε is employed wherever an example involving this parameter appears.

Remark 3.3: If $q = 1$, conditions in (3.14) and (3.21) are the same as those in Theorems 1 in (Márquez et al., 2013a) and (Jaadari et al., 2012), respectively; thus, the latter are particular cases.

Next corollary will show that ANS conditions (Sala and Ariño, 2007) can be seen as a particular case of the results above.

Corollary 3.1: The solution set of conditions (3.25) defined in Theorem 3.3 always include the solution set of LMIs (3.28) presented in Proposition 2 of (Sala and Ariño, 2007) with $\Upsilon_{ij} = A_i P + B_i F_j + (*)$ under the same complexity parameter d.

Proof: From Proposition 2 in (Sala and Ariño, 2007) or Section 2.7:

$$\left(\sum_{i=1}^{r} h_i(z) \right)^d \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) \Upsilon_{ij} < 0, \tag{3.27}$$

and after considering $q = d + 1$, $i_0 = i$ and $i_1 = j$, the conditions to guarantee (3.27) can be rewritten as:

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0 i_1 i_2 \cdots i_q)} \left(A_{i_0} P + B_{i_0} F_{i_1} + P A_{i_0}^T + F_{i_1}^T B_{i_0}^T \right) = \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0 i_1 i_2 \cdots i_q)} \Upsilon_{i_0 i_1} < 0 . \tag{3.28}$$

Recall that $\rho(i_0, i_1, i_2, \cdots, i_q)$ is the set of all permutations of the indexes $i_0, i_1, i_2, \cdots, i_q$. For instance, $\rho(1,1,2) = \{112 \;\; 211 \;\; 121\}$. If $F_{i_1,i_2,\ldots,i_q} = \sum_{i_1 i_2 \cdots i_q \in \rho(i_1, i_2, \ldots, i_q)} F_{i_1}$, $H_{i_1,i_2,\ldots,i_q} = \sum_{i_1 i_2 \cdots i_q \in \rho(i_1, i_2, \ldots, i_q)} P$, and $R_{i_1,i_2,\ldots,i_q} = \sum_{i_1 i_2 \cdots i_q \in \rho(i_1, i_2, \ldots, i_q)} \varepsilon P$ in (3.25) then:

$$\begin{bmatrix} \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} \left(A_{i_0} P + B_{i_0} F_{i_1} + (*) \right) & (*) \\ \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} \varepsilon \left(A_{i_0} P \right)^T & - \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} 2\varepsilon P \end{bmatrix} < 0 . \tag{3.29}$$

Applying Schur complement, (3.29) is equivalent to:

$$\begin{aligned} &\sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} \Upsilon_{i_0 i_1} \\ &+ \frac{1}{2} \varepsilon \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} (A_i P)^T \left(\sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} P \right)^{-1} \sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0, i_1, i_2, \ldots, i_q)} A_i P < 0. \end{aligned} \tag{3.30}$$

Thus, if (3.28) holds, there is always a sufficiently small $\varepsilon > 0$ such that (3.30) — and consequently (3.29) — holds too, thus concluding the proof. ◈

Remark 3.4: As in Corollary 3.1, a similar analysis can be done for conditions in Theorems 3.1 and 3.2 in order to verify the inclusion of results in Proposition 2 of (Sala and Ariño, 2007); they are omitted for brevity.

Remark 3.5: Although the three approaches presented above contain the quadratic result in Proposition 2 of (Sala and Ariño, 2007) as a particular case, it is not possible, to the best of our knowledge, to demonstrate analytically any inclusion among them.

Remark 3.6: d represents the complexity parameter and plays exactly the same role as for (3.6) which relates to "q" as $q = d + 1$.

Comparisons of computational complexity among Theorems 3.1, 3.2, 3.3, and Theorem 5 in (Sala and Ariño, 2007) are presented below. The number of conditions (N_L) required in Theorems 3.1, 3.2, and 3.3 as well as proposition 2 and Theorem 5 in (Sala and Ariño, 2007) are given in Table 3.1 where $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$ and $h_{\max} = floor\left((d+1)/2\right)$.

Table 3.1. Number of LMI conditions (N_L).

Approach	N_L
Theorems 3.1, 3.2, and 3.3 Proposition 2 in (Sala and Ariño, 2007)	$\binom{r+d+1}{d+2}+1$
Theorem 5 in (Sala and Ariño, 2007)	$\sum_{h=0}^{h_{\max}}\binom{r+d+1}{d+2}\binom{r+d+1-2h}{d+2-2h}+EC+1,$ $EC=\begin{cases} r & \text{if d+2 is odd} \\ r(r+1)/2+1 & \text{if d+2 is even}\end{cases}$

On the other hand, the number of decision variables (N_D) for Theorems 3.1, 3.2, and 3.3 as well as Proposition 2 and Theorem 5 in (Sala and Ariño, 2007) are determined in Table 3.2.

Table 3.2. Number of decision variables (N_D).

Approach	N_D
Theorems 3.1 and 3.2	$\frac{n_x}{2}(n_x+1)+(n_xn_u+n_x^2)r^{d+1}$
Theorem 3.3	$\frac{n_x}{2}(n_x+1)+(n_xn_u+2n_x^2)r^{d+1}$
Proposition 2 in (Sala and Ariño, 2007)	$\frac{n_x}{2}(n_x+1)+n_xn_ur$
Theorem 5 in (Sala and Ariño, 2007)	$\frac{n_x}{2}(n_x+1)+n_xn_ur+\sum_{h=1}^{h_{\max}}\frac{n_xr^h}{2}(n_xr^h+1)r^{d+2-2h}+EV$ $EV=\begin{cases} 0 & d+1 \text{ is odd} \\ \frac{n_xr^{h_{\max}+1}}{2}(n_xr^{h_{\max}+1}+1)r^{d+2-2(h_{\max}+1)} & \text{d+1 is even}\end{cases}$

A specific comparison of the number of LMI conditions for the different approaches under consideration is shown in Table 3.3.

Table 3.3. Number of LMI conditions (N_L) and decision variables (N_D).

Parameters	Th. 5 (Sala and Ariño, 2007)		Th. 3.1 and 3.2		Th. 3.3	
	(N_L)	(N_D)	(N_L)	(N_D)	(N_L)	(N_D)
$n_u=1$, $n_x=2$, $d=0$, $r=2$	5	17	4	15	4	23
$n_u=1$, $n_x=2$, $d=1$, $r=3$	14	72	11	57	11	93
$n_u=1$, $n_x=2$, $d=2$, $r=3$	23	369	16	165	16	273
$n_u=2$, $n_x=3$, $d=2$, $r=5$	87	5886	71	1881	71	3006
$n_u=2$, $n_x=3$, $d=3$, $r=8$	921	301878	793	61446	793	98310

It can be seen that Theorem 5 in (Sala and Ariño, 2007) leads quicker to high-size problems and also needs to satisfy more LMI conditions than the other approaches. Another point is the fact that the conditions in Theorem 3.3 require the same number of LMIs, but more decision variables than those presented in Theorems 3.1 and 3.2.

Example 3.2: Consider the following TS model taken from Example 2 in (Sala and Ariño, 2007):

$$\dot{x} = \sum_{i=1}^{3} h_i(z)(A_i x + B_i u), \tag{3.31}$$

where $A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}$.

Selecting $a = 12$, a comparison between conditions in Theorem 3.3, Theorem 5 of (Sala and Ariño, 2007), and (3.28) in Corollary 3.1, is presented in Table 3.4. The parameters to compare are the maximum feasible value of b, the complexity parameter used (d), the number of LMI conditions (N_L) and scalar decision variables (N_D), as well as the time required by the solver to find a solution.

Table 3.4. Comparison with $a = 12$.

Approach	$b_{\max}$	d	N_L	N_D	time(s)
Theorem 3.3	12.3	1	11	93	0.58
Theorem 5 in (Sala and Ariño, 2007)	12.3	2	23	369	2
(3.28) in Corollary 3.1	11.8	200	20707	9	33930

Notice that even with a large value of the complexity parameter $d = 200$ conditions (3.28) in Corollary 3.1 cannot obtain the same $b_{\max}$ as other approaches. The advantages of using conditions (3.25) instead of conditions in Theorem 5 of (Sala and Ariño, 2007) or (3.28) are clear: they significantly lowered the number of LMIs, decision variables (except to (3.28)), and computational time to find a solution.♦

Example 3.1 (continued): Recall that the TS model of Example 3.1 is proved to be stable regardless of the number of its vertices; therefore the goal is to see for the different methods up to what r can be possible to “push”. For a different number of rules, results on applying Theorem 3.3 as well as Theorem 5 in (Sala and Ariño, 2007), are presented in Table 3.5. In this example, conditions in Theorem 3.3 are selected because despite of the fact that they have the same number of LMI conditions with respect to Theorem 3.1 and 3.2, they require more decision variables which bring numerical problems before other approaches.

Table 3.5. Comparison of number of conditions, decisions variables, and time to find a feasible solution for Theorem 3.3 and Theorem 5 in (Sala and Ariño, 2007) with $n_x = 2$, $n_u = 1$, and $q = 3$

	Th. 5 (Sala and Ariño, 2007)			Th. 3.3		
	(N_L)	(N_D)	time(s)	(N_L)	(N_D)	time(s)
$r = 3$	23	369	2	16	273	2
$r = 4$	47	1115	12	36	643	4
$r = 5$	87	2663	54	71	1253	10
$r = 8$	368	16979	4351	331	5123	108
$r = 9$	542	27075	26933	496	7293	282
$r = 10$	772	41123	Failure	716	10003	735
$r = 14$	2487	156635	Failure	2381	27443	11633
$r = 15$	3182	206133	Failure	3061	33753	Failure

Theorem 5 in (Sala and Ariño, 2007) needs more resources searching a feasible solution than Theorem 3.3: in terms of computational time, number of decision variables, and number of conditions to solve. When $r \geq 10$, the solver is unable to find a solution via Theorem 5; whereas for Theorem 3.3, the limit is $r = 15$.♦

Example 3.3: Consider the following TS model:

$$\dot{x} = A_h x + B_h u = \sum_{i=1}^{2} h_i(z)(A_i x + B_i u), \tag{3.32}$$

with $A_1 = \begin{bmatrix} 1 & 5+5a \\ 0 & 10-10b \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 5-5a \\ 0 & 10+10b \end{bmatrix}$, $B_1 = \begin{bmatrix} 1+a \\ 1+b \end{bmatrix}$, $B_2 = \begin{bmatrix} 1-a \\ 1-b \end{bmatrix}$, $\zeta_1 = z_1 = \cos(x_1)$,

$h_1 = \omega_0^1 = \dfrac{1+\cos x_1}{2}$, and $h_2 = 1-\omega_0^1$ defined in the compact set $\mathcal{C}_x = \left\{x : |x_i| \leq \pi/2\right\}$, $i \in \{1,2\}$.

The parameters a and b are varied in the following interval $a \in [-1,1]$ and $b \in [-1,1]$. This example considers $q = 4$ in conditions (3.14), (3.21), and (3.25): all of them provide a larger feasibility region than Theorems 1 in (Jaadari et al., 2012) and (Márquez et al., 2013a), as can be appreciated in Figure 3.1. ♦

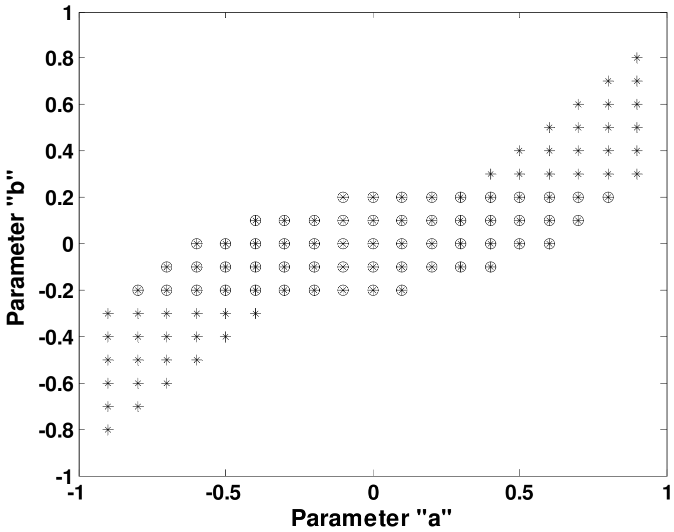


Figure 3.1. Comparison: "*" for (3.14), (3.21), and (3.25) with $q = 4$, "o" for Theorems 1 in (Jaadari et al., 2012) and (Márquez et al., 2013a).

$H\infty$ disturbance attenuation

Consider the case where $w \neq 0$. In order to find LMI conditions that guarantee the system (3.1) under the non-PDC control law (3.2) to hold the $H\infty$ attenuation criterion

$$\sup_{\|w\|_2 \neq 0} \frac{\|y\|_2}{\|w\|_2} \leq \gamma \tag{3.33}$$

for the minimum possible $\gamma > 0$. As shown in (Boyd et al., 1994), condition (3.33) is guaranteed if there exists a Lyapunov functional candidate $V(x)$ such that

$$\dot{V}(x) + \gamma^{-1} y^T y - \gamma w^T w \leq 0. \tag{3.34}$$

Then, using the QLF (3.7), the following inequality is equivalent to (3.34):

$$x^T P^{-1} \dot{x} + \dot{x}^T P^{-1} x + \gamma^{-1} y^T y - \gamma w^T w \leq 0. \tag{3.35}$$

Two options can be used to guarantee $H\infty$ attenuation criterion, both can be obtained considering the closed-loop TS model (3.3) and an extended vector ($\begin{bmatrix} x^T & \dot{x}^T & w^T \end{bmatrix}$ or $\begin{bmatrix} x^T & w^T \end{bmatrix}$):

$$\begin{bmatrix} 0 & P^{-1} & 0 \\ P^{-1} & 0 & 0 \\ 0 & 0 & -\gamma I \end{bmatrix} + \gamma^{-1} \begin{bmatrix} \left(C_h + J_h \mathcal{F}\mathcal{H}^{-1}\right)^T \\ 0 \\ G_h^T \end{bmatrix} \begin{bmatrix} C_h + J_h \mathcal{F}\mathcal{H}^{-1} & 0 & G_h \end{bmatrix} \le 0, \qquad (3.36)$$

or

$$\begin{bmatrix} P^{-1}\left(A_h + B_h \mathcal{F}\mathcal{H}^{-1}\right) + (*) & P^{-1} D_h \\ D_h^T P^{-1} & -\gamma I \end{bmatrix} + \gamma^{-1} \begin{bmatrix} \left(C_h + J_h \mathcal{F}\mathcal{H}^{-1}\right)^T \\ G_h^T \end{bmatrix} \begin{bmatrix} C_h + J_h \mathcal{F}\mathcal{H}^{-1} & G_h \end{bmatrix} \le 0 . \qquad (3.37)$$

As in the first part of this section, similar approaches based on (Shaked, 2001) (Theorem 3.4), (de Oliveira and Skelton, 2001) (Theorem 3.5), and (Peaucelle et al., 2000) (Theorem 3.6) are developed for $H\infty$ controller design. The conditions to satisfy the $H\infty$ attenuation criterion for each approach are given in Table 3.6. These sets of conditions are obtained by applying Schur complement and some properties on matrices to conditions (3.36) or (3.37).

Table 3.6. Conditions of $H\infty$ controller design for different approaches.

Approach	Conditions: $\sum_{i_0 i_1 i_2 \cdots i_q \in \wp\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0$, $\forall\left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1, 2, \ldots, r\}^{q+1}$	Eq.
Theorem 3.4 $J_h = 0$ $\mathcal{F} = F_h$ $\mathcal{H} = H_h$	$\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} -P & \varepsilon D_{i_0} & \varepsilon P C_{i_0}^T & H_{i_1 i_2 \cdots i_q} + \varepsilon\left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q}\right) \\ (*) & -\varepsilon\gamma I & \varepsilon G_{i_0}^T & 0 \\ (*) & (*) & -\varepsilon\gamma I & 0 \\ (*) & (*) & (*) & P - H_{i_1 i_2 \cdots i_q} - H_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$	(3.38)
Theorem 3.5 $\mathcal{F} = F_h$ $\mathcal{H} = H_h$	$\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} + (*) & (*) & (*) & (*) \\ H_{i_1 i_2 \cdots i_q} - P + \varepsilon\left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q}\right) & -2\varepsilon P & (*) & (*) \\ D_{i_0}^T & \varepsilon D_{i_0}^T & -\gamma I & (*) \\ C_{i_0} H_{i_1 i_2 \cdots i_q} + J_{i_0} F_{i_1 i_2 \cdots i_q} & 0 & G_{i_0} & -\gamma I \end{bmatrix}$	(3.39)
Theorem 3.6 $\mathcal{F} = F_h$ $\mathcal{H} = P$	$\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} + (*) & (*) & (*) & (*) \\ D_{i_0}^T & -\gamma I & (*) & (*) \\ C_{i_0} P + J_{i_0} F_{i_1 i_2 \cdots i_q} & G_{i_0} & -\gamma I & (*) \\ P - H_{i_1 i_2 \cdots i_q} + R_{i_1 i_2 \cdots i_q}^T A_{i_0}^T & 0 & 0 & -R_{i_1 i_2 \cdots i_q} - R_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$	(3.40)

Example 3.4: Consider a TS model as in (3.1) with:

$$A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix},\ B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix},\ A_3 = \begin{bmatrix} -5 & -4.33 \\ 0 & 0.05 \end{bmatrix},\ B_3 = \begin{bmatrix} 12 \\ -1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.89 & -5.29 \\ 0.1 & 0 \end{bmatrix},\ B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}^T,\ C_2 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}^T,\ C_3 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}^T,\ C_4 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}^T,$$

$$D_1 = D_3 = \begin{bmatrix} -0.15 \\ 0.3 - 0.1\alpha \end{bmatrix},\ D_2 = D_4 = \begin{bmatrix} -0.15 \\ 0.3 + 0.1\alpha \end{bmatrix},\ J_h = 0,\ G_1 = G_3 = 0.1\alpha,\ G_2 = G_4 = -0.1\alpha,$$

$$r = 4,\ \omega_0^1 = x_1^2,\ \omega_0^2 = \frac{x_2^2}{4},\ \omega_1^1 = 1 - \omega_0^1,\ \omega_1^2 = 1 - \omega_0^2,\ h_1 = \omega_0^1\omega_0^2,\ h_2 = \omega_0^1\omega_1^2,\ h_3 = \omega_1^1\omega_0^2,$$

$h_4 = \omega_1^1\omega_1^2$, where α is a real-valued parameter.

The performance bounds γ obtained by "classical" quadratic approach in (Tuan et al., 2001) as well as conditions in Theorems 3.4, 3.5, and 3.6 for different values of α are provided in Table 3.7 with $q = 1$ and a logarithmically spaced family of values $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$ when it is necessary.

Table 3.7. Comparison of $H\infty$ performances with $q = 1$.

Approach	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
(Tuan et al., 2001)	9.0520	8.5041	7.9715
Theorem 3.4	5.0623	4.7798	4.5063
Theorem 3.5	1.6382	1.5486	1.4616
Theorem 3.6	0.4316	0.4073	0.3907

Table 3.7 shows that the performance level achieved by conditions in Theorem 3.6 is better than results for Theorems 3.4, 3.5, and the quadratic approach. Also, note that conditions (3.40) do not need parameter ε as in (3.38) and (3.39): yet, they still perform better.

Figures 3.2 and 3.3 are presented in order to illustrate the behavior of the attenuation level γ with respect to an increasing parameter q in Theorems 3.4, 3.5, and 3.6. The minimal value for γ is calculated for $\alpha \in [0,1]$.

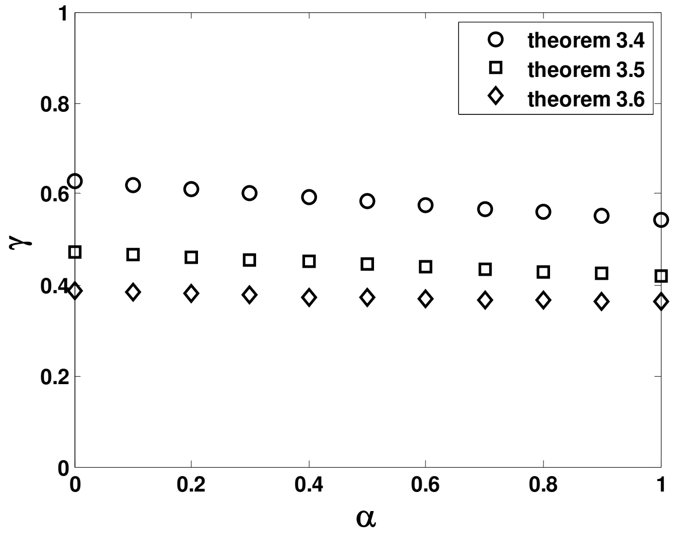


Figure 3.2. γ values for Theorems 3.4, 3.5, and 3.6 with $q = 2$.

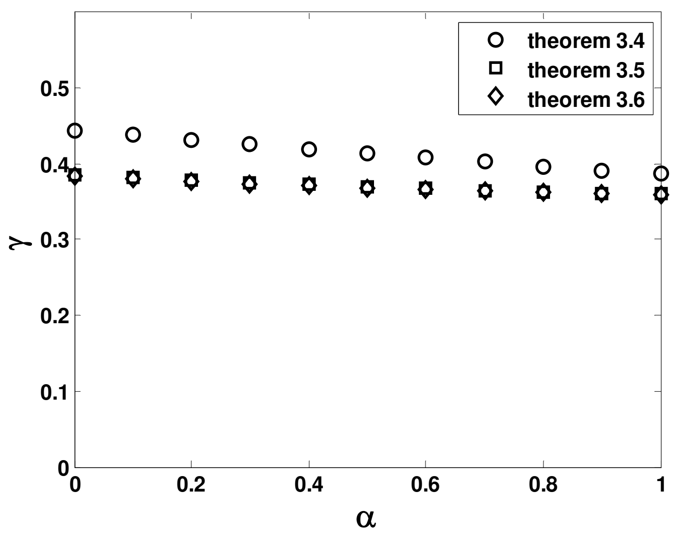


Figure 3.3. γ values for Theorems 3.4, 3.5, and 3.6 with $q = 3$.

It is possible to observe from Figure 3.2 and 3.3 that if parameter q increases, the minimal value of γ decreases altogether with the fact that conditions in Theorem 3.6 always give better results than Theorem 3.4 and 3.5. ♦

Despite the fact that ANS conditions are described in this section, it may occur that quadratic stability is not enough. Many works have already shown this problem and tackled it by using a different sort of Lyapunov functions: (Blanchini, 1995), where polyhedral Lyapunov functions are under consideration, (Zhang and Xie, 2011) using a fuzzy Lyapunov function and homogeneous polynomial techniques. The next section proposes an extension of the previous results to the non-quadratic framework; they will comprise previous works on the subject like those in (Pan et al., 2012); (Lee et al., 2012), and (Jaadari et al., 2012).

3.2.3. Stabilization via fuzzy Lyapunov function

Consider the following fuzzy Lyapunov function (FLF) candidate (Blanco et al., 2001):

$$V(x) = x^T P_h^{-1} x = x^T \left(\sum_{i=1}^{r} h_i(z) P_i \right)^{-1} x, \tag{3.41}$$

where $P_i = P_i^T > 0$, $i \in \{1,\ldots,r\}$.

Results in this section are based on a procedure that "decouples" the control matrices from the Lyapunov function as in (Jaadari et al., 2012). The following generalized non-PDC control law is employed:

$$u = F_{\bar{h}} H_{\bar{h}}^{-1} x, \tag{3.42}$$

which corresponds to $\mathcal{F} = F_{\bar{h}}$ and $\mathcal{H} = H_{\bar{h}}$ in the general form (3.2). Discussion about regularity of $H_{\bar{h}}$ will be done further on. Note that $q = 1$ ($d = 0$) corresponds to the control law structure in (Guerra and Vermeiren, 2004). Then, the state equation of the closed-loop TS model (3.3) without disturbances ($w = 0$) yields as:

$$\dot{x} = \left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1} \right) x. \tag{3.43}$$

It is important to recall the fact that for stabilization via FLF as in (3.41) the term $P_{\dot{h}} = \sum_{k=1}^{r} \dot{h}_k(z) P_k$ appears, which implies a dependency on the time derivatives of the MFs which is difficult to cast as a convex problem. To circumvent this problem, in (Guerra and Bernal, 2009) the following relation is presented (Section 2.4.2):

$$P_{\dot{h}} = \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\partial \omega_0^k}{\partial z_k} \dot{z}_k \left(P_{g_1(i,k)} - P_{g_2(i,k)} \right), \tag{3.44}$$

where $g_1(i,k) = \lfloor (-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (i-1) \bmod 2^{p-k}$ and $g_2(i,k) = g_1(i,k) + 2^{p-k}$, $\lfloor \bullet \rfloor$ stands for the floor function, such that, in order to get LMI conditions, the following bound is stated (Guerra et al., 2011; Pan et al., 2012):

$$\left| \dot{\omega}_0^k \right| \le \beta_k, \ \beta_k > 0, \tag{3.45}$$

with $\dot{\omega}_0^k = \dfrac{\partial \omega_0^k}{\partial z_k} \dot{z}_k$.

Let us begin with an important property called local stabilizability (Theorem 2.7) (Bernal et al., 2010): if conditions in Theorem 2.7 hold then there exists a region containing origin such that it is stable, it means that β_k exists.

For the sake of clarity, the proof for the stabilization problem is split in two: a first part (Lemma 3.1) develops LMI conditions to guarantee that $\left|\dot{\omega}_0^k\right| \le \beta_k$ holds for $\beta_k > 0$, $k \in \{1,2,\ldots,p\}$, where ω_0^k are the weighting functions (WFs); a second part (Theorem 3.7) assumes these bounds are available (since they are guaranteed by the first part) and establishes stability based on them. It is important to stress that both sets of results are run together as a single parameter-dependent LMI problem.

Lemma 3.1: Conditions $\left|\dot{\omega}_0^k\right| \le \beta_k$, $\beta_k > 0$, $k \in \{1,2,\ldots,p\}$ hold for the closed-loop TS model (3.43) if given $\left\|\dfrac{\partial \omega_0^k}{\partial \xi^k}\right\| \le \lambda_k$ and $\|x\| \le \lambda_x$, there exists matrices $Q_{i_1 i_2 \cdots i_q} = Q_{i_1 i_2 \cdots i_q}^T > 0$, $H_{i_1 i_2 \cdots i_q}$, and $F_{i_1 i_2 \cdots i_q}$, $i_1, i_2, \ldots, i_q \in \{1,2,\ldots,r\}$ such that the following conditions hold :

$$\begin{aligned} &\sum_{i_1 i_2 \cdots i_q \in \wp\left(i_1 i_2 \cdots i_q\right)} \Gamma_{i_1 i_2 \cdots i_q} < 0, \quad \forall \left(i_1, i_2, \ldots, i_q\right) \in \{1,2,\ldots,r\}^q \\ &\sum_{i_0 i_1 i_2 \cdots i_q \in \wp\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall \left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1,2,\ldots,r\}^{q+1}, \end{aligned} \tag{3.46}$$

with $\Gamma_{i_1 i_2 \cdots i_q} = Q_{i_1 i_2 \cdots i_q} - \phi_k \left(H_{i_1 i_2 \cdots i_q}^T + H_{i_1 i_2 \cdots i_q} - I\right)$, $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} -\phi_k I & -T_k^T \left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_i F_{i_1 i_2 \cdots i_q}\right) \\ (*) & -Q_{i_1 i_2 \cdots i_q} \end{bmatrix}$,

$\phi_k = \dfrac{2\beta_k}{\lambda_x^2 + \lambda_k^2}$, $\quad T_k = \left[\delta_{ij}^k\right]_{i=1,2,\ldots,n_x;\ j=1,2,\ldots,n_k}$, $\quad \delta_{ij}^k = \begin{cases} 1, & if \quad \dfrac{\partial z_k}{\partial x_i} = \dfrac{\partial z_k}{\partial \xi_j^k} \ne 0 \\ 0, & otherwise \end{cases}$, $\quad n_k \le n_x$,

$\xi^k = \left\{ x_i : \dfrac{\partial z_k}{\partial x_i} \ne 0 \right\} \in \mathbb{R}^{n_k}$.

In order to show how T_k is obtained, consider $z_k = 1 - x_3^2 \cos\left(x_1\right)$ and $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$, then $\frac{\partial z_k}{\partial x_2} = \frac{\partial z_k}{\partial x_4} = 0$ which imply that $\xi^k = \begin{bmatrix} x_1 & x_3 \end{bmatrix}^T$ and $\frac{\partial z_k}{\partial x} = T_k \frac{\partial z_k}{\partial \xi^k}$ with $T_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^T$.

Proof: Each condition $\left|\dot{\omega}_0^k\right| \le \beta_k$, $\beta_k > 0$, $k \in \{1, 2, \ldots, p\}$ can be written as

$$\left|\dot{\omega}_0^k\right| = \left|\frac{\partial \omega_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial x}\right)^T \dot{x}\right| = \left|\frac{\partial \omega_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T T_k^T \left(A_h H_{\bar{h}} + B_h F_{\bar{h}}\right) H_{\bar{h}}^{-1} x\right| \le \beta_k . \tag{3.47}$$

Following the notation in (Pan et al., 2012), a shorthand for previous expression is $\left|\alpha^T \Omega H_{\bar{h}}^{-1} x\right| \le \beta_k$ if $\alpha^T = \frac{\partial \omega_0^k}{\partial z_k}\left(\frac{\partial z_k}{\partial \xi^k}\right)^T$ and $\Omega = T_k^T \left(A_h H_{\bar{h}} + B_h F_{\bar{h}}\right)$. Thus, (3.47) is equivalent to

$$\left|\alpha^T \Omega H_{\bar{h}}^{-1} x\right| + \left|x^T H_{\bar{h}}^{-T} \Omega^T \alpha\right| \le 2\beta_k . \tag{3.48}$$

Applying the property B.5 with $\mathcal{X} = H_{\bar{h}}^{-1} x$ and $\mathcal{Y} = \Omega^T \alpha$ to (3.48) yields the following sufficient condition

$$\alpha^T \Omega Q_{\bar{h}}^{-1} \Omega^T \alpha + x^T H_{\bar{h}}^{-T} Q_{\bar{h}} H_{\bar{h}}^{-1} x \le 2\beta_k , \tag{3.49}$$

which, after considering a priori known bounds $\|\alpha\| \le \lambda_k$ and $\|x\| \le \lambda_x$ gives directly $2\beta_k \left(\|x\|^2 + \|\alpha\|^2\right) \le 2\beta_k \left(\lambda_x^2 + \lambda_k^2\right)$; thus, (3.49) is implied by

$$\begin{bmatrix} x \\ \alpha \end{bmatrix}^T \begin{bmatrix} H_{\bar{h}}^{-T} Q_{\bar{h}} H_{\bar{h}}^{-1} & 0 \\ 0 & \Omega Q_{\bar{h}}^{-1} \Omega^T \end{bmatrix} \begin{bmatrix} x \\ \alpha \end{bmatrix} \le \begin{bmatrix} x \\ \alpha \end{bmatrix}^T \frac{2\beta_k}{\lambda_x^2 + \lambda_k^2} \begin{bmatrix} x \\ \alpha \end{bmatrix},$$

which is equivalent to:

$$\begin{cases} x^T \left(\dfrac{2\beta_k}{\lambda_x^2 + \lambda_k^2} I - H_{\bar{h}}^{-T} Q_{\bar{h}} H_{\bar{h}}^{-1}\right) x \ge 0 \\ \alpha^T \left(\dfrac{2\beta_k}{\lambda_x^2 + \lambda_k^2} I - \Omega Q_{\bar{h}}^{-1} \Omega^T\right) \alpha \ge 0. \end{cases} \tag{3.50}$$

Applying Schur complement to the second expression of (3.50), it renders the following equivalent expression:

$$\begin{bmatrix} \frac{2\beta_k}{\lambda_x^2+\lambda_k^2} I & T_k^T\left(A_h H_{\bar{h}} + B_h F_{\bar{h}}\right) \\ \left(A_h H_{\bar{h}} + B_h F_{\bar{h}}\right)^T T_k & Q_{\bar{h}} \end{bmatrix} \geq 0. \quad (3.51)$$

As for the first expression of (3.50), pre-multiplication by $H_{\bar{h}}^T$ and post-multiplication by $H_{\bar{h}}$ yields the equivalent inequality:

$$\frac{2\beta_k}{\lambda_x^2+\lambda_k^2} H_{\bar{h}}^T H_{\bar{h}} - Q_{\bar{h}} \geq \frac{2\beta_k}{\lambda_x^2+\lambda_k^2}\left(H_{\bar{h}}^T + H_{\bar{h}} - I\right) - Q_{\bar{h}} \geq 0. \quad (3.52)$$

Applying lemma C.4 to (3.51) and (3.52) gives the set of conditions presented in (3.46); thus producing the desired result. ◈

Remark 3.7: Several solutions have appeared to guarantee that $\left|\dot{\omega}_0^k\right| \leq \beta_k$ holds for $\beta_k > 0$, $k \in \{1,2,\ldots,p\}$, under different control schemes (Pan et al., 2012; Lee et al., 2012; Jaadari et al., 2012). Some approaches intend to find the maximum region of attraction while fixing the values of every bound according to their maxima inside the modeling area, which may be as conservative as to preclude any solution on smaller regions still contained in the modeling compact. For this reason, whenever a systematic way to find a non-quadratic solution is found by adjusting the values of bounds as λ_x and λ_k, it will be done in the sequel, for instance, via the bisection method.

Now, using the FLF in (3.41) and the closed-loop TS model (3.43), then, $\dot{V}(x) < 0$ if:

$$P_h^{-1}\left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1}\right) + \left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1}\right)^T P_h^{-1} + P_{\dot{h}}^{-1} < 0. \quad (3.53)$$

Pre- and post-multiplying the previous expression by P_h and because $P_h P_{\dot{h}}^{-1} P_h = -P_{\dot{h}}$, (3.53) is equivalent to:

$$\left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1}\right) P_h + P_h \left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1}\right)^T - P_{\dot{h}} < 0. \quad (3.54)$$

Theorem 3.7: The TS model (3.1) with $w = 0$ under the control law (3.42) is locally asymptotically stable if $\exists \varepsilon > 0$ and matrices $P_{i_1} = P_{i_1}^T > 0$, $H_{i_1 i_2 \cdots i_q}$, and $F_{i_1 i_2 \cdots i_q}$, $i_1, i_2, \ldots, i_q \in \{1,2,\ldots,r\}$ such that the next conditions hold

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \wp\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall \left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1,2,\ldots,r\}^{q+1}, \quad (3.55)$$

$$\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} + (*) - \sum_{k=1}^{p} (-1)^{d_i^\gamma} \beta_k \left(P_{g_1(i_1,k)} - P_{g_2(i_1,k)} \right) & (*) \\ P_{i_1} - H_{i_1 i_2 \cdots i_q} + \varepsilon \left(A_{i_0} H_{i_1 i_2 \cdots i_q} + B_{i_0} F_{i_1 i_2 \cdots i_q} \right)^T & -\varepsilon \left(H_{i_1 i_2 \cdots i_q} + H_{i_1 i_2 \cdots i_q}^T \right) \end{bmatrix},$$

where $g_1(i_1,k) = \lfloor (i_1 - 1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (i_1 - 1) \bmod 2^{p-k}$ and $g_2(i_1,k) = g_1(i_1,k) + 2^{p-k}$, $\lfloor \bullet \rfloor$ stands for the floor function, $\gamma = \{0, \dots, 2^r - 1\}$, $\gamma = 1 + d_r^\gamma + d_{r-1}^\gamma \times 2 + \dots + d_1^\gamma \times 2^{r-1}$, $\left| \dot{\omega}_0^k \right| \le \beta_k$.

Proof: Applying property B.3 to (3.54) with $\mathcal{A}^T = \left(A_h + B_h F_{\bar{h}} H_{\bar{h}}^{-1} \right)$, $\mathcal{L}^T = H_{\bar{h}}$, $\mathcal{R} = \varepsilon H_{\bar{h}}$, $\mathcal{P} = P_h$, and $\mathcal{Q} = -P_{\dot{h}}$, an equivalent inequality is obtained:

$$\begin{bmatrix} A_h H_{\bar{h}} + B_h F_{\bar{h}} + (*) - P_{\dot{h}} & (*) \\ P_h - H_{\bar{h}} + \varepsilon \left(A_h H_{\bar{h}} + B_h F_{\bar{h}} \right)^T & -\varepsilon \left(H_{\bar{h}} + H_{\bar{h}}^T \right) \end{bmatrix} < 0 . \tag{3.56}$$

Since $P_{\dot{h}} = \sum_{k=1}^{p} \left(P_{g_1(h,k)} - P_{g_2(h,k)} \right) \dot{\omega}_0^k$ (Guerra and Bernal, 2009), it is clear that conditions (3.55) guarantee (3.56), thus concluding the proof. ◈

Conditions in Theorem 3.7 are more relaxed than conditions in (Jaadari et al., 2012); a very hard assumption has been employed that consists in the use of the bound $P_{\dot{h}}^{-1} \le \varepsilon P_h^{-1}$. This bound is necessary because $P_{\dot{h}}$ is unsigned and therefore a Schur complement cannot be used for the term $H_h^T P_{\dot{h}}^{-1} H_h$.

Remark 3.8: Regularity of $H_{\bar{h}}$ derives directly from block entry $(2,2)$ in (3.56).

Remark 3.9: Lemma 3.1 altogether with Theorem 3.7 provides conditions for non-quadratic local stabilization of nonlinear systems in the Takagi-Sugeno form. The designed controller guarantee stabilization of the TS model in the outermost level contained in region $\{x : x^T x \le \lambda_x^2\} \cap \mathcal{C}_x$.

Remark 3.10: Recall that conditions in Theorem 3.7 can be run altogether with those in Lemma 3.1, since the latter guarantee that $\left| \dot{\omega}_0^k \right| \le \beta_k$, $k \in \{1, 2, \dots, p\}$. On the other hand, if the property of local stabilizability (Theorem 2.7) holds, and if no feasible solution is found, the bounds in $\|\alpha\| \le \lambda_k$ and $\|x\| \le \lambda_x$ can be gradually reduced in order to find a solution.

Remark 3.11: Note that conditions (3.55) in Theorem 3.7 can be extended to support multi-index Lyapunov functions of the form

$$V(x) = x^T P_{\bar{h}}^{-1} x, \tag{3.57}$$

where $P_{i_1 i_2 \cdots i_q} = P_{i_1 i_2 \cdots i_q}^T > 0$, $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, r\}$, since there are generalizations allowing $P_{\bar{h}}$ to be written in a convex form provided $|\dot{\omega}_0^k| \leq \beta_k$ (Bernal and Guerra, 2010). The set of solutions of Theorem 3.7 via Lyapunov function (3.57) includes that of (Lee and Kim, 2014) because the latter corresponds to the particular choice $H_{\bar{h}} = P_{\bar{h}}$.

Remark 3.12: The number of conditions (N_L) and decision variables (N_D) required for Theorem 3.7 and Theorem 2 in (Lee and Kim, 2014) are given in Table 3.8.

Table 3.8. Number of LMI conditions (N_L) and decision variables (N_D).

Approach	N_L	N_D
Theorem 3.7	$(2^r+1)\binom{r+d+1}{d+2} + \binom{r+d}{d+1} + r$	$\left(\frac{n_x}{2}(n_x+1) + n_x n_u + n_x^2\right) r^{d+1} + \frac{n_x}{2}(n_x+1) r$
Th. 2 (Lee and Kim, 2014)	$(1+p)\binom{r+d}{d+1} + (2^r + rw)\binom{r+d+1}{d+2}$	$(3n_x^2 + n_x + n_x n_u) r^{d+1} + 1$

Example 3.1 (Continued): This example follows the same line as in (Delmotte et al., 2007). In this case the original matrices for the TS model are:

$$A_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.1 & 0 \\ 0.5 & 0 & -0.1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.5 & -1 & 0 \\ -1 & 0.1 & 0 \\ -0.5 & 0 & -0.1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix}, \text{and } B_2 = \begin{bmatrix} 1 \\ 3 \\ 0.5 \end{bmatrix}.$$

Table 3.9. Comparison: number of conditions (N_L), decisions variables (N_D), and time.

	Th. 2 (Lee and Kim, 2014)			Th. 3.7		
	(N_L)	(N_D)	time(s)	(N_L)	(N_D)	time(s)
$q=2$	590	529	45.55	354	312	26.16
$q=3$	1040	2113	268.82	619	1176	187.51
$q=4$	1673	8449	2714	991	4632	1636
$q=5$	2520	33793	Failure	1488	18456	12718

Table 3.9 presents a comparison between Theorem 3.7 and Theorem 2 in (Lee and Kim, 2014); the number of LMI conditions and decisions variables, as well as the time required to find a feasible solution are considered. The following values for comparisons are considered: $n_u = 1$, $n_x = 3$, $p = 2$, $w = 3$, and $r = 4$. From the table above, it can be seen that Theorem 2 in (Lee and Kim, 2014) is not able to solve the problem when $q = 5$ whereas Theorem 3.7 can find a solution. Theorem 3.7 needs less LMI conditions, decision variables, and time to find a feasible solution than Theorem 2 in (Lee and Kim, 2014). In this example (Table 3.9), the following parameters are considered: $|x|_{1,2,3} = 2\pi$, $\phi = 300$ for Theorem 2 in (Lee and Kim, 2014); $|x|_{1,2,3} = 2\pi$, $\beta_{1,2,3} = 300$, and $\varepsilon = 1$ for Theorem 3.7. ♦

Example 3.3 (Continued): Using condition in Theorem 3.7 and selecting $a = 0.8$ and $b = 0.12$, quadratic schemes are unfeasible. Moreover, it is also impossible to find a non-quadratic solution considering the bounds $\lambda_x^2 = \pi^2/2$, $\lambda_k^2 = 0.5^2$, $T_k = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, for any β_k ($k = 1$ in this example), in $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$. Nevertheless property of local stabilizability holds, therefore a solution can be found by gradually reducing certain bounds; for instance, with $\beta_k = 3$, $\lambda_x^2 = 0.1279$, and $\lambda_1^2 = 0.0157$, a non-PDC controller of the form (3.42) can be found via fuzzy Lyapunov function (3.41) for $\varepsilon = 0.01$. Matrices $P_1 = \begin{bmatrix} 35.7358 & 16.6490 \\ 16.6490 & 8.8178 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 39.7234 & 26.9359 \\ 26.9359 & 19.2801 \end{bmatrix}$ define the FLF in the form of (3.41).

Figure 3.4 depicts the biggest stabilization regions with reduced bounds ($\mathcal{C}_{red}$). Two trajectories of the controlled model have been included which show the convergence towards the origin.

In order to verify the bound $|\dot{\omega}_0^k| \le \beta_k$, Figure 3.5 shows the trajectory of $\dot{\omega}_0^1$ with different initial states $x(0)$. It is possible to observe from Figure 3.5 that the bound of $|\dot{\omega}_0^1|$ satisfy the previous assumption $|\dot{\omega}_0^1| \le 3$ under the given initial state.

The reduced region above cannot be obtained with conditions in Theorem 2 of (Pan et al., 2012), but with $\beta_k = 3$, $\lambda_x^2 = 0.0747$, and $\lambda_k^2 = 0.0092$, this approach is feasible. Note, however, that it is smaller than the region achieved with conditions in Theorem 3.7. ♦

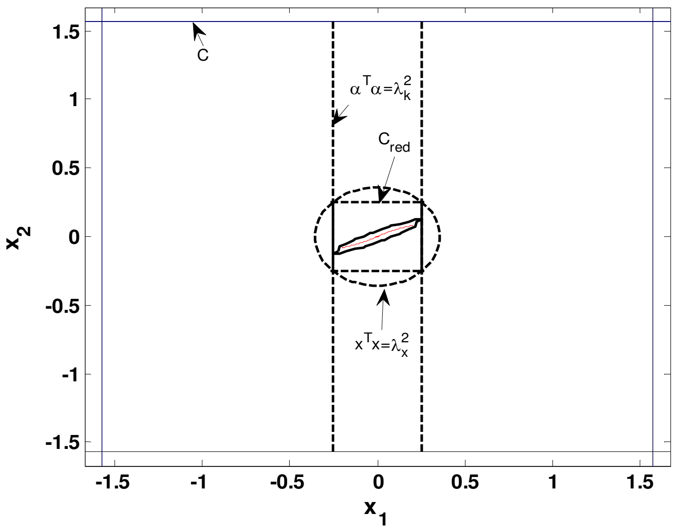


Figure 3.4. Reduced stabilization region under Theorem 3.7 scheme.

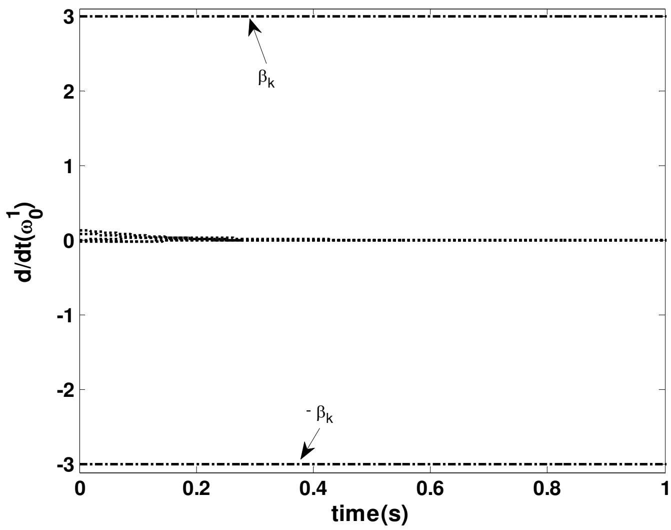


Figure 3.5. Bound β_k.

Example 3.5: Consider the following 4-rule TS model:

$$\dot{x} = A_h x + B_h u = \sum_{i=1}^{4} h_i(z)(A_i x + B_i u), \tag{3.58}$$

with

$$A_1 = \begin{bmatrix} -3 & 2 \\ 0 & 0.9 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.8 & 3 \\ 0 & -0.9 \end{bmatrix},\ A_3 = \begin{bmatrix} -1.9 & 2 \\ -0.5 & 0.1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0.1 & 3 \\ 0 & -2 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad B_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\quad \zeta_1 = z_1 = x_1^2,\quad \zeta_2 = z_2 = x_2^2,\quad \omega_0^1 = \frac{4 - x_1^2}{4},\quad \omega_0^2 = \frac{4 - x_2^2}{4},\quad \omega_1^1 = 1 - \omega_0^1,$$

$\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1 \omega_0^2$, $h_2 = \omega_0^1 \omega_1^2$, $h_3 = \omega_1^1 \omega_0^2$, $h_4 = \omega_1^1 \omega_1^2$ in the compact set $\mathcal{C}_x = \left\{ x : \left| x_j \right| \le 2 \right\}$, $j = 1, 2$.

The non-quadratic stabilization conditions proposed in (Bernal and Guerra, 2010) for the TS model (3.58) are based on direct bounds over the time-derivatives of MFs and the input control law , i.e. $\left| \frac{\partial \omega_0^k}{\partial x_v} x_s \right| \le \lambda_{kvs}$, $\left| \frac{\partial w_0^k}{\partial x_v} \right| \le \eta_{kv}$, and $\left| u(t) \right| \le \mu$; these bounds have to be taken into account to validate the stabilization region. On the other hand, Theorem 3.7 establishes a local approach where bounds of $R = \left\{ x : x^T x \le \lambda_x^2 \right\}$ and the modeling region $\mathcal{C}_x$ are already taken into account and validated via LMIs in (3.46). Considering $\lambda_x^2 = 8$, $\lambda_k^2 = 1$, $T_k = I$, and $\beta_k = 20$ ($k = 1, 2$), with $\varepsilon = 0.01$, a feasible solution with $q = 3$ can be found; the FLF thus is given by (3.41) with $P_1 = \begin{bmatrix} 1.0924 & 0.6338 \\ 0.6338 & 1.6431 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.8443 & -0.0719 \\ -0.0719 & 0.5968 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1.0926 & 0.6331 \\ 0.6331 & 1.6440 \end{bmatrix}$, and $P_4 = \begin{bmatrix} 0.8455 & -0.0733 \\ -0.0733 & 0.5966 \end{bmatrix}$.

Figure 3.6 compares the stabilization domains of results in (3.55) (R_1) with those in (Bernal and Guerra, 2010) (R_0) for the TS model (3.58): it is clear that the new approach presents the biggest domain of attraction. ♦

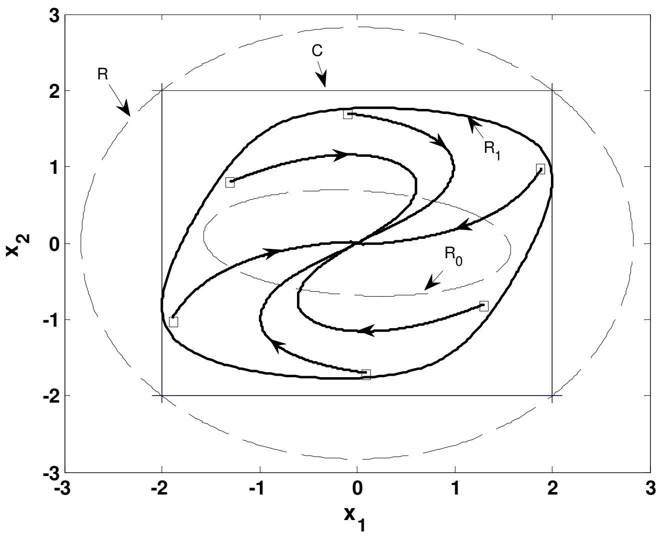


Figure 3.6. Lyapunov level for (3.55) (R_1) and conditions in (Bernal and Guerra, 2010) (R_0).

3.2.4. Stabilization via line-integral Lyapunov function

An alternative to circumvent the time-derivative of MFs when non-quadratic Lyapunov functions are used (which lies behind *local* conditions appearing in the literature) has been proposed in (Rhee and Won, 2006); it is based on a line-integral Lyapunov function (Khalil, 2002) which gives *global* conditions inside the compact set of the state space. An extension of this approach is developed below; its scope being restricted to second-order systems.

Let us consider the following line-integral Lyapunov function candidate (Rhee and Won, 2006):

$$V(x) = 2\int_{\Gamma(0,x)} \mathfrak{F}(\psi)\, d\psi \,. \tag{3.59}$$

As mentioned before in Section 2.3.3., a special structure on $\mathfrak{F}(x)$ has been proposed in order to satisfy the path-independency condition:

$$\mathfrak{F}(x) = \left(\sum_{i=1}^{r} h_i(x)\left(\bar{P} + \mathfrak{D}_i\right)\right) x = P(x)\, x\,, \tag{3.60}$$

with

$$\bar{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n_x} \\ p_{12} & 0 & \cdots & p_{2n_x} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n_x} & p_{2n_x} & \cdots & 0 \end{bmatrix}, \quad \mathfrak{D}_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n_x n_x}^{\alpha_{in_x}} \end{bmatrix}, \quad h_i(x) = \prod_{j=1}^{n_x} \omega_j^{\alpha_{ij}}(x_j) \quad \text{where}$$

$\omega_j^{\alpha_{ij}}(x_j)$ are the WFs, and $\bar{P} + \mathfrak{D}_i = \left(\bar{P} + \mathfrak{D}_i\right)^T > 0$.

The conditions for stabilization presented in (Rhee and Won, 2006) are BMI; therefore, they are not optimally solvable because existing methods may lead to local minima.

Now, consider $\mathfrak{F}(x) = P(x)^{-1} x$, where $P(x)^{-1} = \sum_{i=1}^{r} h_i(x) P_i$, $P_i = \bar{P} + \mathfrak{D}_i > 0$ a symmetric matrix, the time-derivative of the Lyapunov function in (3.59) is

$$\dot{V}(x) = L_g V(x) = \mathfrak{F}^T(x)\, g(x) + g^T(x)\, \mathfrak{F}(x) = x^T P(x)^{-T} \dot{x} + \dot{x}^T P(x)^{-1} x\,, \tag{3.61}$$

where $g(x) = \dot{x}$; using (3.3) with $w = 0$ then (3.61) yields:

$$\dot{V}(x) = x^T \left(P(x)^{-T} \left(A_h + B_h \mathcal{F}\mathcal{H}^{-1} \right) + \left(A_h + B_h \mathcal{F}\mathcal{H}^{-1} \right)^T P(x)^{-1} \right) x\,. \tag{3.62}$$

To guarantee $\dot{V}(x) < 0$, then

$$P(x)^{-T}\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right) + \left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T P(x)^{-1} < 0. \tag{3.63}$$

Multiplying by $P(x)^T$ on the right-hand side and by its transpose $P(x)$ on the left-hand side of (3.63), gives

$$\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)P(x) + P(x)^T\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T < 0. \tag{3.64}$$

Considering that from (Rhee and Won, 2006) we know that for a 4-rule 2nd-order system, the Lyapunov function is path independent if and only if:

$$P(x)^{-1} = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}, \tag{3.65}$$

where $d_{11}^{\alpha_{i1}} = \omega_0^1\overline{d}_{11} + \left(1-\omega_0^1\right)\underline{d}_{11}$, $d_{22}^{\alpha_{i2}} = \omega_0^2\overline{d}_{22} + \left(1-\omega_0^2\right)\underline{d}_{22}$, and $\overline{d}_{11}$, $\underline{d}_{11}$ $\overline{d}_{22}$, $\underline{d}_{22}$, and q are constants, then we can obtain the inverse directly

$$P(x) = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}^{-1} = \frac{1}{d_{11}^{\alpha_{i1}}d_{22}^{\alpha_{i2}} - q^2}\begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}, \tag{3.66}$$

and rewrite (3.66) as

$$P(x) = \frac{X_h}{\left|P(x)^{-1}\right|}, \tag{3.67}$$

with $X_h = \begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}$ and $\left|P(x)^{-1}\right| = d_{11}^{\alpha_{i1}}d_{22}^{\alpha_{i2}} - q^2$.

Now, substituting (3.67) in (3.64):

$$\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)\frac{X_h}{\left|P(x)^{-1}\right|} + \left(\frac{X_h}{\left|P(x)^{-1}\right|}\right)^T\left(A_h + B_h\mathcal{F}\mathcal{H}^{-1}\right)^T < 0,$$

which can be simplified as

$$\frac{1}{\left|P(x)^{-1}\right|}\left(A_hX_h + B_h\mathcal{F}\mathcal{H}^{-1}X_h + X_hA_h^T + X_h\mathcal{H}^{-T}\mathcal{F}^TB_h^T\right) < 0. \tag{3.68}$$

Due to the fact that $\frac{1}{\left|P(x)^{-1}\right|} > 0$, then (3.68) can be written as

$$A_h X_h + B_h \mathcal{F}\mathcal{H}^{-1} X_h + X_h A_h^T + X_h \mathcal{H}^{-T} \mathcal{F}^T B_h^T < 0, \tag{3.69}$$

which leads to the following theorem.

Theorem 3.8. The TS model (3.1) with MF's $h_i(x)$ as in (3.60) and $w=0$ under the control law (3.2) is globally asymptotically stable if there exist matrices $X_i = X_i^T > 0$ and F_i such that the next LMI conditions are satisfied:

$$\begin{aligned} &\Upsilon_{ii} < 0, \ \forall i \in \{1,2,\cdots,r\} \\ &\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \ \forall (i,j) \in \{1,2,\cdots,r\}^2, \ i \neq j, \end{aligned} \tag{3.70}$$

with $\Upsilon_{ij} = A_i X_j + B_i F_j + X_j A_i^T + F_j^T B_i^T$.

Proof: Assuming $\mathcal{H} = X_h$ and $\mathcal{F} = F_h$ in (3.69), it yields

$$A_h X_h + B_h F_h + X_h A_h^T + F_h^T B_h^T < 0. \tag{3.71}$$

Applying the relaxation lemma C.3 to the previous expression gives the desired result, thus concluding the proof.◈

Table 3.10. Conditions of controller design for new approaches.

Approach	Control law $u = \mathcal{F}\mathcal{H}^{-1}x$	Conditions: $\begin{cases} \Upsilon_{ii} < 0, \ i \in \{1,2,\cdots,r\}, \\ \frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \ i,j \in \{1,2,\cdots,r\}^2, \ i \neq j. \end{cases}$	Eq.
Theorem 3.9 (Tustin-like)	$\mathcal{F} = F_h$ $\mathcal{H} = H_h$	$\Upsilon_{ij} = \begin{bmatrix} -X_i & H_j + \varepsilon(A_i H_j + B_i F_j) \\ (*) & X_i - H_j - H_j^T \end{bmatrix}$	(3.72)
Theorem 3.10 (Finsler)	$\mathcal{F} = F_h$ $\mathcal{H} = H_h$	$\Upsilon_{ij} = \begin{bmatrix} A_i H_j + B_i F_j + (A_i H_j + B_i F_j)^T & (*) \\ H_j - X_i + \varepsilon(A_i H_j + B_i F_j) & -2\varepsilon X_i \end{bmatrix}$	(3.73)
Theorem 3.11 (Peaucelle)	$\mathcal{F} = F_h$ $\mathcal{H} = X_h$	$\Upsilon_{ij} = \begin{bmatrix} A_i H_j + B_i F_j + (A_i H_j + B_i F_j)^T & (*) \\ X_i - H_j + R_j^T A_i^T & -R_j - R_j^T \end{bmatrix}$	(3.74)

Remark 3.13: The suggested $P(x)$ in (3.66) is only valid for second-order systems; higher order systems lead to products of decision variables which cannot be treated as a convex problem.

The refinements shown in Section 3.2.2, which are based on matrix transformations, can also be applied to the previous result in order to obtain more relaxed conditions; these are shown in Table 3.10: they can be developed from (3.61) following the same paths of Section 3.2.2 for each approach.

Example 3.6: Consider the following T-S model (Example 2 in (Rhee and Won, 2006)):

$$\dot{x} = \sum_{i=1}^{2} h_i(z)(A_i x + B_i u), \tag{3.75}$$

with $A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} a & -5 \\ 1 & 2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} b \\ 2 \end{bmatrix}$, $a \in [-20, 10]$, and $b \in [0, 25]$; the normalized MF are given by $h_1(z) = \begin{cases} (1 - \sin x_1)/2 & \text{for } |x_1| \le \pi/2, \\ 0 & \text{for } |x_1| > \pi/2 \\ 1 & \text{for } |x_1| < -\pi/2 \end{cases}$, $h_2(z) = 1 - h_1(z)$.

Figure 3.7 shows that the feasibility set of conditions (3.72) includes that of (Rhee and Won, 2006).

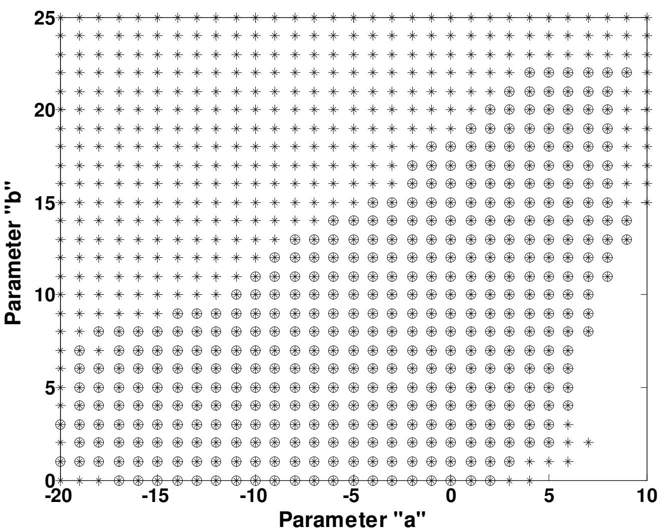


Figure 3.7. Comparison: "$*$" for (3.72) and "$\circ$" for conditions in (Rhee and Won, 2006).

Let us illustrate a particular case. Selecting $a = 4$, $b = 1$, and $\varepsilon = 0.1$, a non-PDC controller as in (3.72) can be found via a line-integral Lyapunov function candidate (3.59). The gains and Lyapunov matrices are given by:

$P_1^{-1} = \begin{bmatrix} 496.3 & -759 \\ -759 & 1217.5 \end{bmatrix}$, $P_2^{-1} = \begin{bmatrix} 502.2 & -759 \\ -759 & 1217.5 \end{bmatrix}$, $H_1 = \begin{bmatrix} 835.9 & 561.7 \\ 393.9 & 315.3 \end{bmatrix}$, $H_2 = \begin{bmatrix} 777.2 & 216.5 \\ 799 & 559.2 \end{bmatrix}$,

$F_1 = \begin{bmatrix} -4996 & -5190 \end{bmatrix}$, and $F_2 = \begin{bmatrix} -4363 & -4772 \end{bmatrix}$.

Figure 3.8 shows that states are effectively stabilized; initial conditions $x(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ were considered. ♦

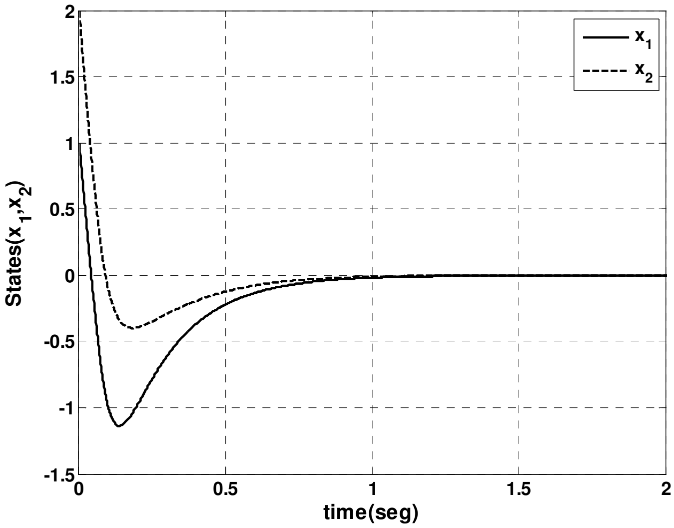


Figure 3.8. Time evolution of the states.

3.2.5. Stabilization via non-quadratic Lyapunov functional

This section proposes a new Lyapunov functional that includes the quadratic framework as a particular case, while solving the problem of the time-derivatives of the MFs. As the one proposed in (Rhee and Won, 2006), it produces *global* instead of *local* conditions, but it does so without imposing the tough restrictions for path-independent line integrals and without any limitation on the system order.

Consider the following non-quadratic Lyapunov functional (NQLF):

$$V(x) = x^T P_s^{-1} x = x^T \left(\sum_{i=1}^{r} s_i\big(z(t)\big) P_i \right)^{-1} x, \tag{3.76}$$

with $P_i = P_i^T > 0$, and

$$s_i\big(z(t)\big) = \frac{1}{\alpha} \int_{t-\alpha}^{t} h_i\big(z(\tau)\big) d\tau \geq 0, \; \alpha > 0. \tag{3.77}$$

It is important to stress the fact that MFs $h_i(\cdot)$ are integrable along the trajectories because they are smooth and bounded.

The MFs $s_i(\cdot)$, $i \in \{1,2,\ldots,r\}$ in (3.77), inherit the convex sum property from the model MFs $h_i(\cdot)$:

$$\sum_{i=1}^{r} s_i(z(t)) = \frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\sum_{i=1}^{r} h_i(z(\tau))\right)d\tau = 1. \tag{3.78}$$

Notice also that the time-derivative of MFs s_i is:

$$\dot{s}_i(z) = \frac{1}{\alpha}\big(h_i(z(t)) - h_i(z(t-\alpha))\big), \tag{3.79}$$

with $x(t) = \phi(t)$, $t \in [-\alpha, 0]$, $\phi \in \ell([-\alpha,0],\mathbb{R})$ being the initial function and $\ell([-\alpha,0],\mathbb{R})$ the Banach space of real continuous functions on the interval $[-\alpha,0]$ with $|\phi|_\alpha = \max_{t\in[-\alpha,0]}|\phi(\theta)|$ (Gu et al., 2003).

Therefore:

$$P_{\dot{s}} = \frac{1}{\alpha}\left(P_h - P_{h^-}\right),\ h_i^- = h_i(z(t-\alpha)). \tag{3.80}$$

For any fixed parameter $\alpha > 0$, it follows that $V(0) = 0$ and $V(x) > 0$. Moreover, with $\underline{\lambda} = \left(\max_{k\in\{1,2,\ldots,r\}}(\lambda_{P_k})\right)^{-1}$ and $\overline{\lambda} = \left(\min_{k\in\{1,2,\ldots,r\}}(\lambda_{P_k})\right)^{-1}$ where λ_{P_k} are the eigenvalues of P_k, and due to the convex sum property, the following holds:

$$\underline{\lambda}\|x\|^2 \le x^T P_s^{-1} x \le \overline{\lambda}\|x\|^2, \tag{3.81}$$

which proves that $V(x)$ is a valid Lyapunov functional candidate.

Remark 3.14: The proposed NQLF (3.76) is inspired by the discrete-time case which exhibits former samples of the MFs $h(z(t-1))$. Convexity in (3.77) as well as the fact that $\lim_{\alpha\to 0}\dot{s}_i(z) = \lim_{\alpha\to 0}\frac{1}{\alpha}\big(h_i(z(t)) - h_i(z(t-\alpha))\big) = \dot{h}_i(z(t))$ give additional consistency to the approach.

A non-PDC control law, introducing the α-delay of the Lyapunov functional is adopted:

$$u = F_{hh^- s} P_s^{-1} x, \tag{3.82}$$

where $F_{hh^- s} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i h_j^- s_k F_{ijk}$, $P_s = \sum_{k=1}^{r} s_k P_k$, $F_{ijk} \in \mathbb{R}^{n_u \times n_x}$, and $P_k \in \mathbb{R}^{n_x \times n_x}$, $i, j, k \in \{1, 2, \ldots, r\}$ are matrices of proper size related with the notion of controller gains.

Then, the closed-loop TS model writes

$$\dot{x} = \left(A_h + B_h F_{hh^- s} P_s^{-1}\right) x + D_h w, \tag{3.83}$$

which, in the absence of perturbations, i.e. $w = 0$, yields

$$\dot{x} = \left(A_h + B_h F_{hh^- s} P_s^{-1}\right) x. \tag{3.84}$$

Consider the NQLF candidate in (3.76) with $P_i = P_i^T > 0$, $\forall i \in \{1, 2, \ldots, r\}$ altogether with the closed-loop model (3.84); $\dot{V}(x) < 0$ is satisfied if:

$$P_s^{-1}\left(A_h + B_h F_{hh^- s} P_s^{-1}\right) + \left(A_h + B_h F_{hh^- s} P_s^{-1}\right)^T P_s^{-1} + P_{\dot{s}}^{-1} < 0. \tag{3.85}$$

Multiplying left and right by P_s, the previous expression yields

$$A_h P_s + B_h F_{hh^- s} + \left(A_h P_s + B_h F_{hh^- s}\right)^T + P_s P_{\dot{s}}^{-1} P_s < 0, \tag{3.86}$$

from which the next rewriting can be done with $P_s P_{\dot{s}}^{-1} P_s = -P_{\dot{s}}$:

$$A_h P_s + B_h F_{hh^- s} + \left(A_h P_s + B_h F_{hh^- s}\right)^T - P_{\dot{s}} < 0. \tag{3.87}$$

Substituting (3.80) in (3.87), it yields:

$$A_h P_s + B_h F_{hh^- s} + \left(A_h P_s + B_h F_{hh^- s}\right)^T - \frac{1}{\alpha}\left(P_h - P_{h^-}\right) < 0. \tag{3.88}$$

Theorem 3.12. The TS model (3.1) with $w = 0$ under the control law (3.82) is globally asymptotically stable if there exist a scalar $\alpha > 0$ and matrices $P_i = P_i^T > 0$, F_{jkl}, $i, j, k, l \in \{1, 2, \ldots, r\}$, such that the following conditions are satisfied

$$\begin{aligned} &\Upsilon_{ii}^{kl} < 0,\ \forall (i,k,l) \in \{1,2,\cdots,r\}^3, \\ &\frac{2}{r-1}\Upsilon_{ii}^{kl} + \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl} < 0,\ \forall (i,j,k,l) \in \{1,2,\cdots,r\}^4,\ i \neq j, \end{aligned} \tag{3.89}$$

with $\Upsilon_{ij}^{kl} = A_i P_l + B_i F_{jkl} + (*) - \frac{1}{\alpha}(P_i - P_k)$.

Proof: Let us check first the existence of P_s^{-1}. As $\forall i,\ P_i > 0$ and the MFs $s_i(z(t)) \ge 0$ having a convex sum property, $P_s > 0$ for every $s(t)$ therefore P_s^{-1} exists. Now, applying the relaxation lemma C.5 to (3.88) leads to conditions (3.89), which concludes the proof. ◈

Corollary 3.2: Under the same relaxation conditions, the solution set for LMIs (3.89) with $\Upsilon_{ij}^{kl} = \Upsilon_{ij} = A_i P + B_i F_j + (*)$, which corresponds to the solution set of a quadratic Lyapunov function with a PDC control law (Tanaka and Wang, 2001), is included in the solution set of (3.89) in Theorem 3.12.

Proof: It follows directly if $P_s = P = P^T > 0$ and $F_{hh^- s} = F_h$ in (3.88). ◈

More general results can be found via property B.3, which conveniently allows an α-dependent formulation with slack matrices that significantly increase the feasibility area.

Theorem 3.13. The TS model (3.1) with $w = 0$ under the control law (3.82) is globally asymptotically stable if there exists a scalar $\alpha > 0$ and matrices $P_i = P_i^T > 0$, F_{jkl}, H_{jkl}, R_{jkl}, $i, j, k, l \in \{1, 2, \ldots, r\}$, such that the following constraints are satisfied:

$$
\begin{aligned}
&\Upsilon_{ii}^{kl} < 0,\ \forall (i,k,l) \in \{1,2,\cdots,r\}^3, \\
&\frac{2}{r-1}\Upsilon_{ii}^{kl} + \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl} < 0,\ \forall (i,j,k,l) \in \{1,2,\cdots,r\}^4,\ i \ne j,
\end{aligned}
\tag{3.90}
$$

with $\Upsilon_{ij}^{kl} = \begin{bmatrix} A_i H_{jkl} + B_i F_{jkl} + (*) - \frac{1}{\alpha}(P_j - P_k) & (*) \\ P_l - H_{jkl} + R_{jkl}^T A_i^T & -R_{jkl} - R_{jkl}^T \end{bmatrix}$.

Proof: The time-derivative of the NQLF (3.76) holds $\dot{V}(x) < 0$ if:

$$
A_h P_s + B_h F_{hh^- s} + \left(A_h P_s + B_h F_{hh^- s}\right)^T - P_{\dot{s}} < 0. \tag{3.91}
$$

Applying property B.3 with $\mathcal{A} = A_h^T$, $\mathcal{L} = H_{hh^- s}^T$, $\mathcal{P} = P_s$, $\mathcal{R} = R_{hh^- s}$, $\mathcal{Q} = B_h F_{hh^- s} + F_{hh^- s}^T B_h^T - P_{\dot{s}}$ and $P_{\dot{s}} = \frac{1}{\alpha}\left(P_h - P_{h^-}\right)$, the next inequality arises:

$$\begin{bmatrix} A_h H_{hh^-s} + B_h F_{hh^-s} + \left(A_h H_{hh^-s} + B_h F_{hh^-s}\right)^T - \frac{1}{\alpha}\left(P_h - P_{h^-}\right) & (*) \\ P_s - H_{hh^-s} + R_{hh^-s}^T A_h^T & -R_{hh^-s} - R_{hh^-s}^T \end{bmatrix} < 0. \quad (3.92)$$

The use of lemma C.5 on the previous inequality gives conditions (3.90), concluding the proof. ◈

Corollary 3.3: The solution set for LMIs (3.89) defined in Theorem 3.12 is included in the solution set of (3.90) defined in Theorem 3.13.

Proof: The solution set of (3.90) of Theorem 3.13 guarantees (3.92). Pre- and post-multiplying (3.92) by $\begin{bmatrix} I & A_h \end{bmatrix}$ and its transpose, respectively produces (3.88), which is guaranteed by LMIs (3.89) of Theorem 3.12, thus concluding the proof. ◈

Remark 3.15: Theorem 3.13 significantly increases the number of decision variables. If needed, a way to reduce it is to assume that $F_{hh^-s} = F_h$ while preserving the advantages of the proposed approach.

Table 3.11 compares the number of LMIs rows (N_L) and scalar decision variables (N_D) of Theorem 3.12, Theorem 3.13, and Theorem 3.13 with $F_{hh^-s} = F_h$ (as in Remark 3.15), where n_x, n_u, and r are the number of states, inputs, and rules, respectively. It is clear that Theorem 3.13 in any of its two cases needs more decision variables and twice the number of LMI conditions than Theorem 3.12. Note also that despite the fact that the number of LMIs for both versions of Theorem 3.13 is the same, there is an important difference with respect to the number of decision variables involved: a less complex control law ($F_{hh^-s} = F_h$) needs less decision variables.

Table 3.11. Number of LMI rows (N_L) and scalar decision variables (N_D) in the conditions.

Approach	N_L	N_D
Theorem 3.12	$n_x\left(r^4 + r\right)$	$\frac{n_x}{2}\left(n_x + 1\right)r + n_x n_u r^3$
Theorem 3.13	$n_x\left(2r^4 + r\right)$	$\frac{n_x}{2}\left(n_x + 1\right)r + n_x n_u r^3 + 2n_x^2 r^3$
Theorem 3.13 - Remark 3.15	$n_x\left(2r^4 + r\right)$	$\frac{n_x}{2}\left(n_x + 1\right)r + n_x n_u r + 2n_x^2 r^3$

Example 3.7: Consider the following TS model:

$$\begin{aligned} \dot{x} &= A_h x + B_h u + D_h w = \sum_{i=1}^{4} h_i(z)\left(A_i x + B_i u + D_i w\right) \\ y &= C_h x + J_h u + G_h w = \sum_{i=1}^{4} h_i(z)\left(C_i x + J_i u + G_i w\right), \end{aligned} \tag{3.93}$$

with $A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0.89 & -5.29 \\ 0.1 & 0 \end{bmatrix}$,

$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} -b+6 \\ -1 \end{bmatrix}$, $B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}^T$, $C_3 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}^T$,

$C_4 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}^T$, $J_1 = J_3 = 0.1 - 0.05\eta$, $J_2 = J_4 = 0.1 + 0.05\eta$, $G_1 = G_3 = 0.1\eta$, $G_2 = G_4 = -0.1\eta$,

$D_1 = D_3 = \begin{bmatrix} -0.15 \\ 0.1 - 0.05\eta \end{bmatrix}$, $D_2 = D_4 = \begin{bmatrix} -0.15 \\ 0.1 + 0.05\eta \end{bmatrix}$, $\omega_0^1 = \frac{1 - \sin(x_1)}{2}$, $\omega_0^2 = \frac{4 - x_2^2}{4}$, $\omega_1^1 = 1 - \omega_0^1$,

$\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1 \omega_0^2$, $h_2 = \omega_0^1 \omega_1^2$, $h_3 = \omega_1^1 \omega_0^2$, $h_4 = \omega_1^1 \omega_1^2$, and η being a real-valued parameter.

In order to compare the feasibility sets of conditions in Theorems 3.12 and 3.13 with the quadratic case in (Tanaka and Wang, 2001) as well as Theorem 1 in (Jaadari et al., 2012), no disturbances are considered in this example, i.e., $w = 0$. As expected from the inclusions analytically proven in Corollaries 3.1 and 3.2, Figure 3.9 shows that the feasibility region corresponding to LMI constraints (3.90) (Theorem 3.13) overcomes that corresponding to (3.89) (Theorem 3.12), which in turn includes the feasibility set corresponding to the quadratic approaches in (Tanaka and Wang, 2001) and (Jaadari et al., 2012) (based on Finsler transformation).

Similar comparisons are shown in Figure 3.10 between the feasibility set of Theorem 3.13 and those of conditions in Theorem 7 of (Mozelli et al., 2009) and Theorem 3.9, which have been chosen among non-quadratic schemes for being global as the approach hereby proposed. It is clear that larger feasibility regions can be found with the new approach when compared with results in (Mozelli et al., 2009) and Theorem 3.9. Moreover, Figure 3.10 also illustrates that conditions in (Mozelli et al., 2009) are not able to find solutions due to its special structure which enforces path-independency of the line integral.

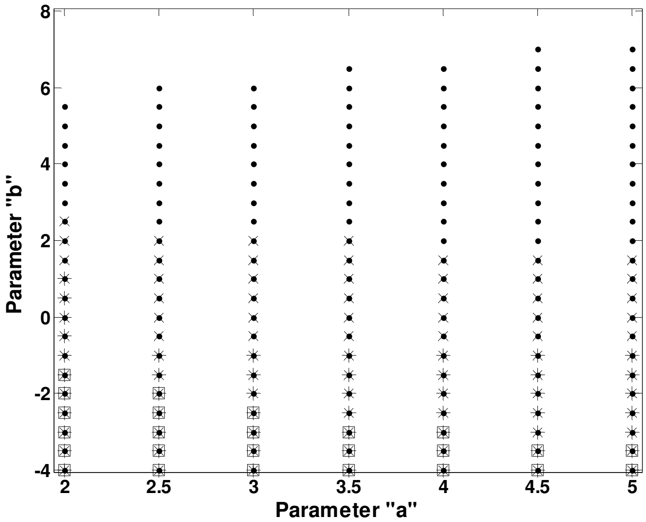


Figure 3.9. Comparison: "$\bullet$" for Th. 3.13, "$\times$" for Th. 3.12, "$+$" for Th. 1 in (Jaadari et al., 2012), and "$\square$" for quadratic case in (Tanaka and Wang, 2001).

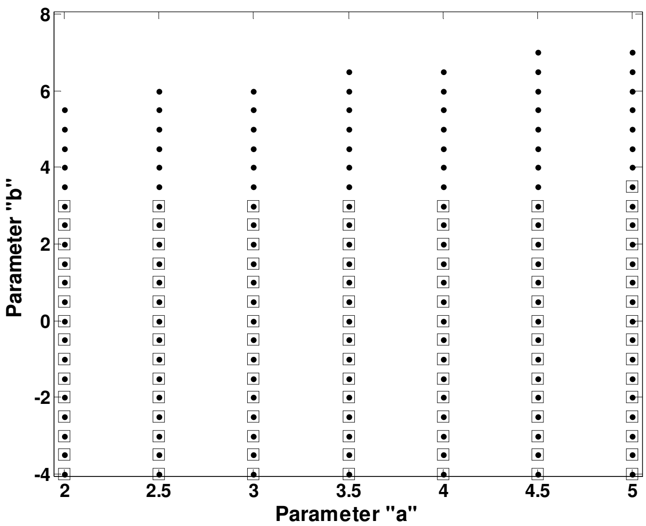


Figure 3.10. Comparison: "$\bullet$" for Th. 3.13, "$\times$" for Th. 7 in (Mozelli et al., 2009), and "$\square$" for Th. 3.9.

In order to illustrate a particular case, the following values have been selected: system parameters $a=3$ and $b=4$, time delay $\alpha=0.1$s. Note that the resulting matrices A_i, $i\in\{1,2,3,4\}$ are all unstable. A controller of the form (3.82) was found via a non-quadratic Lyapunov functional (3.76) through conditions in Theorem 3.13. Due to space limitations, the Lyapunov matrices and some of the 64 gains F_{jkl}, $j,k,l\in\{1,2,3,4\}$ in (3.82) are given for illustration purposes:

$$P_1 = \begin{bmatrix} 15.6517 & -3.3324 \\ -3.3324 & 1.3307 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6.6858 & -2.0481 \\ -2.0481 & 1.3326 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 9.5808 & -2.0659 \\ -2.0659 & 1.2911 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 12.8732 & -2.7964 \\ -2.7964 & 1.3189 \end{bmatrix}, \quad F_{111} = \begin{bmatrix} -132.5205 \\ 11.5539 \end{bmatrix}^T, \quad F_{222} = \begin{bmatrix} 8.7844 \\ 0.9082 \end{bmatrix}^T, \quad F_{333} = \begin{bmatrix} -5.6030 \\ 1.8348 \end{bmatrix}^T,$$

$$F_{444} = \begin{bmatrix} -44.1904 \\ 4.7248 \end{bmatrix}^T.$$

Figure 3.11 shows the time response of the Lyapunov functional (a) and the states (b) from initial condition $x(0) = [0.7 \quad -0.5]$; as expected, the Lyapunov functional decreases monotonically and the states are driven towards the origin.

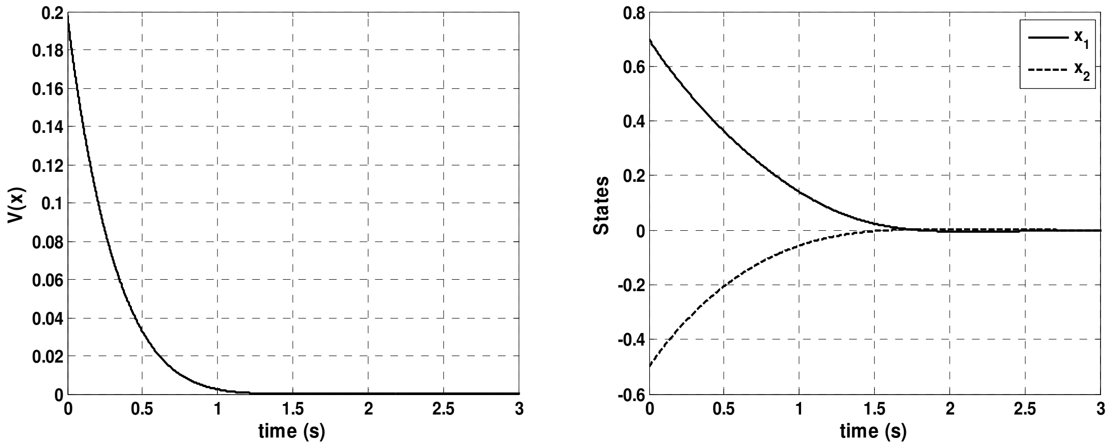


Figure 3.11. Non-perturbed time evolution with $u = F_{hh^-s} P_s^{-1} x$: a) $V(x)$ (left); b) states (right).

The results above can lead to problems in practical terms when a complex control law as in (3.82) is used due to the necessity of calculating several multiplications and performing matrix inversion, especially in nonlinear TS representations with a high number of rules. However, it is still possible to relax this problem reducing the complexity in the proposed controller (3.82) by considering $F_{hh^-s} = F_h$ (Remark 3.15). If such a reduced control law is used, i.e., a controller of the form $u = F_h P_s^{-1} x$, the Lyapunov matrices and gains are:

$$P_1 = \begin{bmatrix} 17.3466 & -2.4356 \\ -2.4356 & 1.2183 \end{bmatrix}, \ F_1 = \begin{bmatrix} -120.9134 \\ 9.8819 \end{bmatrix}^T, \ P_2 = \begin{bmatrix} 14.0184 & -2.2833 \\ -2.2833 & 1.2098 \end{bmatrix}, \ F_2 = \begin{bmatrix} -50.6415 \\ 0.1977 \end{bmatrix}^T,$$

$$P_3 = \begin{bmatrix} 17.6680 & -2.2991 \\ -2.2991 & 1.1723 \end{bmatrix}, \ F_3 = \begin{bmatrix} 14.0533 \\ 4.9980 \end{bmatrix}^T, \ P_4 = \begin{bmatrix} 17.2673 & -2.5281 \\ -2.5281 & 1.1954 \end{bmatrix}, \text{ and } F_4 = \begin{bmatrix} -73.0781 \\ 5.3767 \end{bmatrix}^T.$$

The number of LMI rows for both cases is $N_L = 1032$ and the number of decision variables is $N_D = 652$ and $N_D = 532$ for Theorem 3.13 and Remark 3.15, respectively. Note that despite the reduced complexity of the control law, a feasible solution is achieved. Even with this "reduced" version, it is possible to obtain better results than former approaches. Simulations in Figure 3.12 exhibit the behavior on time of the Lyapunov function (a) and the states (b). ♦

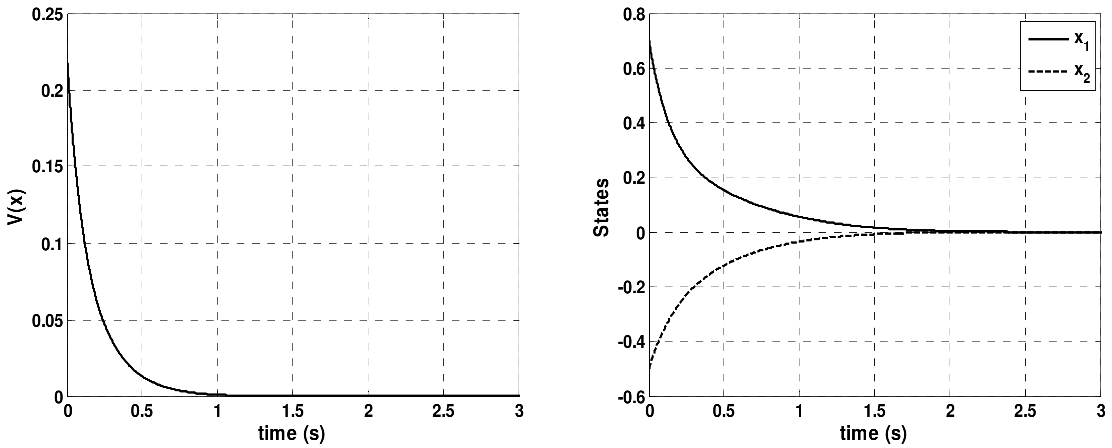


Figure 3.12. Time response without disturbance and $u = F_h P_s^{-1} x$: a) $V(x)$ (left); b) states (right).

$H\infty$ disturbance rejection

Now, consider the case where $w \neq 0$. As in Section 3.2.2, the TS model (3.1) satisfies the $H\infty$ attenuation criterion $\gamma > 0$ if there exists a Lyapunov functional candidate $V(x)$ such that the following well-known condition holds (Boyd et al., 1994):

$$\dot{V}(x) + \gamma^{-1} y^T y - \gamma w^T w \leq 0. \tag{3.94}$$

Then, using the NQLF (3.76), the following inequality is equivalent to (3.94):

$$x^T P_s^{-1} \dot{x} + \dot{x}^T P_s^{-1} x + x^T P_{\dot{s}}^{-1} x + \gamma^{-1} y^T y - \gamma w^T w \leq 0, \tag{3.95}$$

which can be rearranged as follows once the TS model (3.1) under the non-PDC control law (3.82) are taken into account:

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left\{ \begin{array}{c} \begin{bmatrix} P_s^{-1}\left(A_h + B_h F_{hh^-s} P_s^{-1}\right) + (*) + P_{\dot{s}}^{-1} & P_s^{-1} D_h \\ D_h^T P_s^{-1} & -\gamma I \end{bmatrix} \\ + \gamma^{-1} \begin{bmatrix} \left(C_h + J_h F_{hh^-s} P_s^{-1}\right)^T \\ G_h^T \end{bmatrix} \begin{bmatrix} \left(C_h + J_h F_{hh^-s} P_s^{-1}\right) & G_h \end{bmatrix} \end{array} \right\} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0. \tag{3.96}$$

Applying Schur complement to the block-matrix in the middle of the last expression, it can be seen that the inequality therein is guaranteed if

$$\begin{bmatrix} P_s^{-1}\left(A_h + B_h F_{hh^-s} P_s^{-1}\right) + (*) + P_{\dot{s}}^{-1} & P_s^{-1} D_h & C_h^T + P_s^{-1} F_{hh^-s}^T J_h^T \\ D_h^T P_s^{-1} & -\gamma I & G_h^T \\ C_h + J_h F_{hh^-s} P_s^{-1} & G_h & -\gamma I \end{bmatrix} < 0,$$

which after pre- and post-multiplying by block-diag $\left(P_s, I, I\right)$ and taking into account that $P_s P_{\dot{s}}^{-1} P_s = -P_{\dot{s}}$, it is equivalent to

$$\begin{bmatrix} A_h P_s + B_h F_{hh^-s} + (*) - P_{\dot{s}} & D_h & P_s C_h^T + F_{hh^-s}^T J_h^T \\ D_h^T & -\gamma I & G_h^T \\ C_h P_s + J_h F_{hh^-s} & G_h & -\gamma I \end{bmatrix} < 0. \tag{3.97}$$

Then, the next theorem is derived.

Theorem 3.14. The TS model (3.1) under the control law (3.82) is globally asymptotically stable with disturbance attenuation γ if there exist a scalar $\alpha > 0$ and matrices $P_j = P_j^T > 0$, F_{jkl}, H_{jkl}, R_{jkl}, $j,k,l \in \{1,2,\ldots,r\}$ such that the following conditions are satisfied:

$$\begin{aligned} &\Upsilon_{ii}^{kl} < 0,\ \forall(i,k,l) \in \{1,2,\cdots,r\}^3, \\ &\frac{2}{r-1}\Upsilon_{ii}^{kl} + \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl} < 0,\ \forall(i,j,k,l) \in \{1,2,\cdots,r\}^4,\ i \neq j, \end{aligned} \tag{3.98}$$

with $\Upsilon_{ij}^{kl} = \begin{bmatrix} A_i H_{jkl} + B_i F_{jkl} + (*) - \frac{1}{\alpha}\left(P_j - P_k\right) & (*) & (*) & (*) \\ P_l - H_{jkl} + R_{jkl}^T A_i^T & -R_{jkl} - R_{jkl}^T & (*) & (*) \\ E_i^T & 0 & -\gamma I & (*) \\ C_i P_l + D_i F_{jkl} & 0 & G_i & -\gamma I \end{bmatrix}$.

Proof: As before (Theorem 3.13), property B.3 can be applied to the block entry $(1,1)$ in (3.97) with $\mathcal{A} = A_h^T$, $\mathcal{L} = H_{hh^-s}^T$, $\mathcal{R} = R_{hh^-s}$, $\mathcal{P} = P_s$, $\mathcal{Q} = B_h F_{hh^-s} + (*) - P_{\dot{s}}$, and $P_{\dot{s}} = \frac{1}{\alpha}\left(P_h - P_{h^-}\right)$, as to obtain the following sufficient condition:

$$\begin{bmatrix} A_h H_{hh^- s} + B_h F_{hh^- s} + (*) - \frac{1}{\alpha}\left(P_h - P_{h^-}\right) & (*) & (*) & (*) \\ P_s - H_{hh^- s} + R^T_{hh^- s} A^T_h & -R_{hh^- s} - R^T_{hh^- s} & (*) & (*) \\ E^T_h & 0 & -\gamma I & (*) \\ C_h P_s + D_h F_{hh^- s} & 0 & G_h & -\gamma I \end{bmatrix} < 0 . \tag{3.99}$$

Now, applying relaxation lemma C.5 to (3.99) leads to conditions (3.98), which concludes the proof. ◈

Remark 3.16: Note that a simpler solution preserving the use of the proposed NQLF (3.76) as well as the non-PDC control law (3.82) can be obtained from a direct application of lemma C.5 as to guarantee (3.97) with $P_{\dot{s}} = \frac{1}{\alpha}\left(P_h - P_{h^-}\right)$ and

$$\Upsilon^{kl}_{ij} = \begin{bmatrix} A_i P_l + B_i F_{jkl} + (*) - \frac{1}{\alpha}\left(P_j - P_k\right) & (*) & (*) \\ E^T_i & -\gamma I & (*) \\ C_i P_l + D_i F_{jkl} & G_i & -\gamma I \end{bmatrix} . \tag{3.100}$$

The solution set for LMIs in lemma C.5 with Υ^{kl}_{ij} defined as in (3.100) is included in the solution set of (3.98) defined in Theorem 3.14.

Example 3.7 (Continued): Consider the TS model in (3.93) with $w \neq 0$, system parameters $a = 5$ and $b = -6$, and time delay $\alpha = 1$s. The minimum performance bound γ such that (3.94) holds via quadratic conditions in (Tanaka and Wang, 2001), Theorem 3.14 , and conditions (3.100) in Remark 3.16, is provided in Table 3.12 for different values of η .

Table 3.12. Comparison of $H\infty$ performances.

Approach	$\eta = -1$	$\eta = 0$	$\eta = 1$
(Tanaka and Wang, 2001)	88.9285	56.3613	27.5164
Remark 3.16	63.2932	40.1710	20.1161
Theorem 3.14	44.4336	29.1970	13.3715

Once again, the inclusions discussed in Remark 3.16 are verified. Table 3.12 shows that the best $H\infty$ attenuation criterion is obtained via conditions in Theorem 3.1, followed by those in Remark 3.16, which in turn are better than those corresponding to the quadratic approach. Figure 3.13 is presented in order to illustrate the behavior of the $H\infty$ attenuation value γ with respect to parameter η for conditions in Theorem 3.14, Remark 3.16, and the quadratic approach in (Tanaka and Wang, 2001). The minimal value for γ is calculated for equally-spaced values in $\eta \in [-1,1]$. The ability of the proposed approach to reach lower minima for γ is thus illustrated.

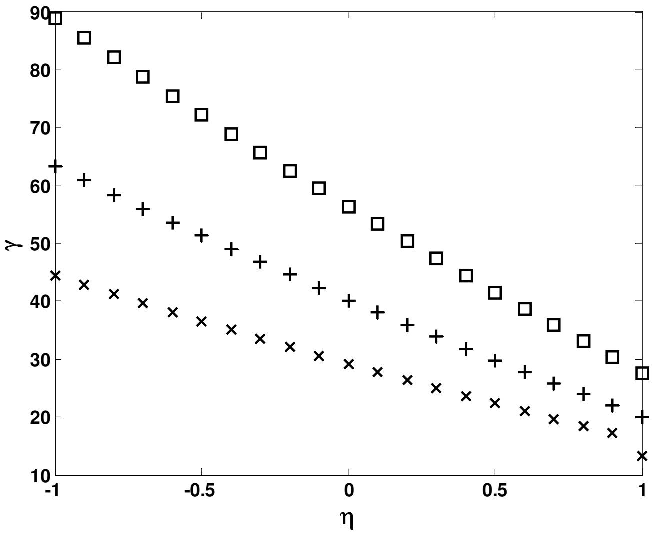


Figure 3.13. γ values: "□" for (Tanaka and Wang, 2001), "+" for Remark 3.16, and "×" for Theorem 3.14.

Figure 3.14 illustrates the effect over the minimal value of γ (z axis) of parameters η and α (axis on the floor plane); the same plot is shown from two different angles in order to appreciate the effect of the parameters. Note that the best results (lowest minima) are obtained via Theorem 3.14, as expected. ♦

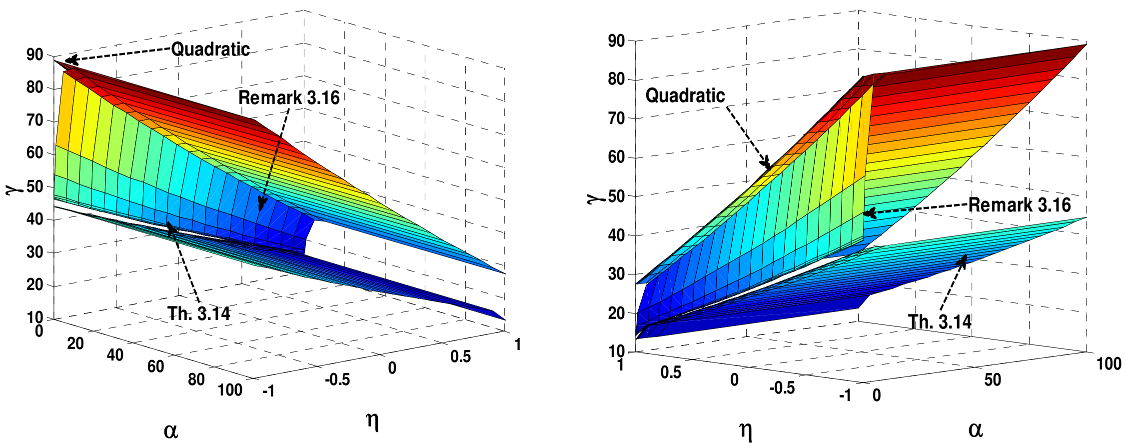


Figure 3.14. γ-η-α behavior for Theorem 3.14, Remark 3.16, and conditions in (Tanaka and Wang, 2001).

Example 3.8: Consider the nonlinear model of a flexible joint robot (Banks et al., 2005):

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{mgl}{I_1}\sin(x_1) - \frac{k}{I_1}(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{I_2}u - \frac{k}{I_2}(x_3 - x_1),
\end{aligned} \tag{3.101}$$

where u is the torque input, I_1 is the link inertia, I_2 is the motor inertia, m is the mass, g is the gravity, l is the link length, k is the stiffness, x_1 and x_3 are angular positions of first and second joints respectively.

From (3.101) a TS representation considering perturbations in the model can be obtained as follows:

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{2} h_i(z)\left(A_i x + B_i u + D_i w\right) \\
y &= \sum_{i=1}^{2} h_i(z)\left(C_i x + J_i u + G_i w\right),
\end{aligned} \tag{3.102}$$

with $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{I_1}\left(k + mgl\underline{\zeta}\right) & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{I_1}\left(k + mgl\overline{\zeta}\right) & 0 & \frac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_2} & 0 & -\frac{k}{I_2} & 0 \end{bmatrix}$,

$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_2} \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -290 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 290 \end{bmatrix}$, $J_1 = -0.02$, $J_2 = 0.2$,

$G_1 = 4$, $G_2 = -0.67$, $\zeta(z_1) = z_1 = \frac{\sin(x_1)}{x_1}$, $\underline{\zeta} = \min \zeta$, $\overline{\zeta} = \max \zeta$, $|x_1| \le 2\pi$, $h_1 = \frac{\overline{\zeta} - \zeta(z_1)}{\overline{\zeta} - \underline{\zeta}}$,

$h_2 = 1 - h_1$. Matrices D_i, J_i, and G_i $i \in \{1,2\}$ assumed to be random such that a comparison of disturbance attenuation with (Tanaka and Wang, 2001) can be done.

Under the following model parameters $I_1 = I_2 = 1\, kg \cdot m^2$, $m = 1kg$, $k = 10\, N \cdot m \cdot rad^{-1}$, $l = 1m$, and $g = 9.81 m \cdot s^{-2}$, a non-PDC controller $u = F_h P_s^{-1}$ is designed using conditions (3.98) in Theorem 3.14 with $F_{hh^-s} = F_h$ and $\alpha = 0.1s$; the Lyapunov matrices and gains are given by:

$$P_1 = \begin{bmatrix} 0.0002 & -0.0002 & 0.0001 & 0.0002 \\ -0.0002 & 0.0019 & -0.0006 & 0 \\ 0.0001 & -0.0006 & 0.0002 & -0.0002 \\ 0.0002 & 0 & -0.0002 & 0.0016 \end{bmatrix},\ F_1 = \begin{bmatrix} 0 & 0.0034 & 0 & -7089 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0002 & -0.0003 & 0.0001 & 0.0002 \\ -0.0003 & 0.0020 & -0.0006 & 0 \\ 0.0001 & -0.0006 & 0.0003 & -0.0002 \\ 0.0002 & 0 & -0.0002 & 0.0018 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.001 & 0.001 & 0 & -1450 \end{bmatrix},$$

with an attenuation value of $\gamma = 58.6$ ($\gamma = 59.2$ for (Tanaka and Wang, 2001)).

Notice that our solution for this example improves the disturbance attenuation level with respect to conditions in (Tanaka and Wang, 2001), thus illustrating the effectiveness of the result in Theorem 3.14. ♦

3.3. State feedback controller design for descriptor TS models

In this section the problem of state-feedback controller design for TS models in a descriptor form is addressed. Two schemes will be proposed: the first one is based on a more general non-PDC control law in a semi-quadratic framework, with a straightforward extension to $H\infty$ disturbance rejection; the second one is based on a line-integral fuzzy Lyapunov function and a non-PDC control law. Both schemes give more relaxed conditions than former approaches as it is shown via illustrative examples.

3.3.1. Problem statement

Consider the following continuous-time TS model in the descriptor form:

$$\begin{aligned} E_v \dot{x} &= A_h x + B_h u + D_h w \\ y &= C_h x + G_h w, \end{aligned} \tag{3.103}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$, and $w \in \mathbb{R}^{n_w}$ are the state vector, the control input, the output vector, and the disturbance vector, respectively; the sums $A_h = \sum_{i=1}^{r} h_i(z) A_i$,

$B_h = \sum_{i=1}^{r} h_i(z) B_i$, $C_h = \sum_{i=1}^{r} h_i(z) C_i$, $D_h = \sum_{i=1}^{r} h_i(z) D_i$, $G_h = \sum_{i=1}^{r} h_i(z) G_i$, and $E_v = \sum_{k=1}^{r_e} v_k(z) E_k$ depend on matrices of appropriate dimensions A_i, B_i, D_i, C_i $i \in \{1,2,\ldots,r\}$, and E_k, $k \in \{1,2,\ldots,r_e\}$. The two sets of MFs $0 \le h_i(z) \le 1$, $i \in \{1,2,\ldots,r\}$ and $0 \le v_k(z) \le 1$, $k \in \{1,2,\ldots,r_e\}$ hold the convex sum property $\sum_{i=1}^{r} h_i(z) = 1$ and $\sum_{k=1}^{r_e} v_k(z) = 1$ in a compact set of the state variables; they depend on a premise vector $z \in \mathbb{R}^p$ which in turn depends on the state x.

Remark 3.17: This chapter considers descriptors (3.103) with E_v being a regular matrix for every $z \in \mathbb{R}^p$, i.e., a classical TS could be obtained considering $E_v^{-1}(z)$ but with a possible increase of complexity (see Example 2.4). Therefore the goal therein is to keep a descriptor TS structure and to derive conditions from it.

As stated before in Section 2.8, a shorthand notation, using the extended vector $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$, for the descriptor TS model (3.103) can be written as:

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \bar{A}_{hv}\bar{x} + \bar{B}_h u + \bar{D}_h w \\ y &= \bar{C}_h \bar{x} + G_h w, \end{aligned} \tag{3.104}$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h = \begin{bmatrix} 0 \\ B_h \end{bmatrix}$, $\bar{D}_h = \begin{bmatrix} 0 \\ D_h \end{bmatrix}$, and $\bar{C}_h = \begin{bmatrix} C_h & 0 \end{bmatrix}$.

In the following developments, some alternatives will be proposed to reduce conservativeness of previous approaches such as (Taniguchi et al., 2000; Guerra et al., 2007; Bouarar et al., 2010; Estrada-Manzo et al., 2013).

3.3.2. Stabilization via quadratic Lyapunov function

Consider the following quadratic Lyapunov function candidate (Guerra et al., 2007; Taniguchi et al., 2000):

$$V(\bar{x}) = \bar{x}^T \bar{E}^T \bar{P}_{hhv}^{-1} \bar{x}, \ \bar{E}^T \bar{P}_{hhv}^{-1} = \bar{P}_{hhv}^{-T} \bar{E}, \tag{3.105}$$

with $\bar{P}_{hhv} = \begin{bmatrix} P^1 & 0 \\ P_{hhv}^{21} & P_{hhv}^{22} \end{bmatrix}$, $\bar{P}_{hhv}^{-1} = \begin{bmatrix} \left(P^1\right)^{-1} & 0 \\ -\left(P_{hhv}^{22}\right)^{-1} P_{hhv}^{21} \left(P^1\right)^{-1} & \left(P_{hhv}^{22}\right)^{-1} \end{bmatrix}$, $P_{hhv}^{21} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} P_{ijk}^{21}$ as a free matrix, $P_{hhv}^{22} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} P_{ijk}^{22}$ being a regular matrix, and $P^1 = \left(P^1\right)^T > 0$. Note that

P^1 has been chosen as a constant matrix in order to avoid the time-derivatives of the MFs; this guarantees $\bar{E}^T \dot{\bar{P}}_{hhv}^{-1} = 0$ in the following developments. Normally, the control law is $u = F_{hv}^1 \left(P^1\right)^{-1} x = \begin{bmatrix} F_{hv}^1 & 0 \end{bmatrix} \begin{bmatrix} P^1 & 0 \\ P_{hhv}^{21} & P_{hhv}^{22} \end{bmatrix}^{-1} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ (Taniguchi et al., 2000; Guerra et al., 2007), in order that the control law is not dependent on $\dot{x}$. In the following, the proposal is to extend $\begin{bmatrix} F_{hv}^1 & 0 \end{bmatrix}$ to $\begin{bmatrix} F_{hv}^1 & F_{hv}^2 \end{bmatrix}$, such that the non-PDC control law writes:

$$u = \begin{bmatrix} F_{hv}^1 & F_{hv}^2 \end{bmatrix} \begin{bmatrix} P^1 & 0 \\ P_{hhv}^{21} & P_{hhv}^{22} \end{bmatrix}^{-1} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \bar{F}_{hv} \bar{P}_{hhv}^{-1} \bar{x} , \tag{3.106}$$

with $F_{hv}^1 = \sum_{j=1}^{r} \sum_{k=1}^{r_e} h_j v_k F_{jk}^1$ and $F_{hv}^2 = \sum_{j=1}^{r} \sum_{k=1}^{r_e} h_j v_k F_{jk}^2$ including MFs of both sides of the descriptor model (h_i and v_k), and gains F_{jk}^1, F_{jk}^2, $j \in \{1, 2, \ldots, r\}$, $k \in \{1, 2, \ldots, r_e\}$ to be determined. The feasibility of (3.106) will be discussed further on.

Remark 3.18: The non-PDC control law (3.106) is composed by the states and their time-derivatives in the following form:

$$u = \begin{bmatrix} F_{hv}^1 & F_{hv}^2 \end{bmatrix} \begin{bmatrix} \left(P^1\right)^{-1} & 0 \\ -\left(P_{hhv}^{22}\right)^{-1} P_{hhv}^{21} \left(P^1\right)^{-1} & \left(P_{hhv}^{22}\right)^{-1} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \mathcal{F}_1 x + \mathcal{F}_2 \dot{x} , \tag{3.107}$$

with $\mathcal{F}_1 = F_{hv}^1 \left(P^1\right)^{-1} - F_{hv}^2 \left(P_{hhv}^{22}\right)^{-1} P_{hhv}^{21} \left(P^1\right)^{-1}$ and $\mathcal{F}_2 = F_{hv}^2 \left(P_{hhv}^{22}\right)^{-1}$.

The closed-loop TS descriptor model is obtained after substituting (3.106) in (3.104):

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \left(\bar{A}_{hv} + \bar{B}_h \bar{F}_{hv} \bar{P}_{hhv}^{-1}\right)\bar{x} + \bar{D}_h w \\ y &= \bar{C}_h \bar{x} + G_h w. \end{aligned} \tag{3.108}$$

If no disturbances ($w = 0$) are considered, then the closed-loop descriptor TS model yields:

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \left(\bar{A}_{hv} + \bar{B}_h \bar{F}_{hv} \bar{P}_{hhv}^{-1}\right)\bar{x} \\ y &= \bar{C}_h \bar{x}. \end{aligned} \tag{3.109}$$

The time-derivative of the Lyapunov function candidate in (3.105) is:

$$\dot{V}\left(\bar{x}\right) = \bar{x}^T \bar{E}^T \bar{P}_{hhv}^{-1} \dot{\bar{x}} + \dot{\bar{x}}^T \bar{E}^T \bar{P}_{hhv}^{-1} \bar{x} + \bar{x}^T \bar{E}^T \dot{\bar{P}}_{hhv}^{-1} \bar{x} ,$$

and because $\bar{E}^T \bar{P}_{hhv}^{-1} = \bar{P}_{hhv}^{-T} \bar{E}$, $\bar{E}\dot{\bar{P}}_{hhv}^{-1} = 0$, $\dot{V}\left(\bar{x}\right) < 0$ can be guaranteed if

$$\overline{x}^T \overline{P}_{hhv}^{-T} \overline{E}\dot{\overline{x}} + \dot{\overline{x}}^T \overline{E}^T \overline{P}_{hhv}^{-1} \overline{x} < 0 . \tag{3.110}$$

Using the state equation without disturbances in (3.109) as well as the congruence property with $\overline{P}_{hhv}^T$, the previous expression holds if:

$$\overline{A}_{hv} \overline{P}_{hhv} + \overline{B}_h \overline{F}_{hv} + \left(\overline{A}_{hv} \overline{P}_{hhv} + \overline{B}_h \overline{F}_{hv} \right)^T < 0 . \tag{3.111}$$

Then, the following theorem is stated.

Theorem 3.15: The TS descriptor model (3.103) with $w(t) = 0$ under the control law (3.106) is asymptotically stable if there exist a symmetric and positive definite matrix P^1 and matrices P_{ijk}^{21}, P_{ijk}^{22}, F_{jk}^1, F_{jk}^2, H_{jk}^{11}, H_{jk}^{12}, H_{ij}^{21}, H_{ij}^{22}, R_{jk}^{11}, R_{jk}^{12}, R_{ij}^{21}, R_{ij}^{22}, $i, j \in \{1, 2, \ldots, r\}$, $k \in \{1, 2, \ldots, r_e\}$, such that the next LMI conditions hold:

$$\begin{aligned} &\Upsilon_{ii}^k < 0, \ \forall i \in \{1, 2, \cdots, r\}, \ \forall k \in \{1, 2, \cdots, r_e\} \\ &\frac{2}{r-1} \Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \ \forall (i, j) \in \{1, 2, \cdots, r\}^2, \ i \neq j, \ \forall k \in \{1, 2, \cdots, r_e\}, \end{aligned} \tag{3.112}$$

with $\Upsilon_{ij}^k = \begin{bmatrix} \Gamma_{ij}^{11} & (*) & (*) & (*) \\ \Gamma_{ijk}^{21} & \Gamma_{ijk}^{22} & (*) & (*) \\ \Gamma_{ijk}^{31} & \Gamma_{ijk}^{32} & \Gamma_{jk}^{33} & (*) \\ \Gamma_{ijk}^{41} & \Gamma_{ijk}^{42} & \Gamma_{ijk}^{43} & \Gamma_{ij}^{44} \end{bmatrix}$, $\Gamma_{ij}^{11} = H_{ij}^{21} + \left(H_{ij}^{21} \right)^T$, $\Gamma_{ijk}^{21} = A_i H_{jk}^{11} + B_i F_{jk}^1 - E_k H_{ij}^{21} + \left(H_{ij}^{22} \right)^T$,

$\Gamma_{ijk}^{22} = A_i H_{jk}^{12} + B_i F_{jk}^2 - E_k H_{ij}^{22} + (*)$, $\Gamma_{ijk}^{31} = P^1 - H_{jk}^{11} + \left(R_{ij}^{21} \right)^T$, $\Gamma_{ijk}^{32} = -H_{jk}^{12} + \left(A_i R_{jk}^{11} - E_k R_{ij}^{21} \right)^T$,

$\Gamma_{jk}^{33} = -R_{jk}^{11} - \left(R_{jk}^{11} \right)^T$, $\Gamma_{ijk}^{41} = P_{ijk}^{21} - H_{ij}^{21} + \left(R_{ij}^{22} \right)^T$, $\Gamma_{ijk}^{42} = P_{ijk}^{22} - H_{ij}^{22} + \left(A_i R_{jk}^{12} - E_k R_{ij}^{22} \right)^T$,

$\Gamma_{ijk}^{43} = -R_{ij}^{21} - \left(R_{jk}^{12} \right)^T$, $\Gamma_{ij}^{44} = -R_{ij}^{22} - \left(R_{ij}^{22} \right)^T$.

Moreover, if the conditions are satisfied the control law writes:

$$u = \left(I - \mathcal{F}_2 E_v^{-1} B_h \right)^{-1} \left(\mathcal{F}_1 + \mathcal{F}_2 E_v^{-1} A_h \right) x , \tag{3.113}$$

with $\mathcal{F}_1 = F_{hv}^1 \left(P^1 \right)^{-1} - F_{hv}^2 \left(P_{hhv}^{22} \right)^{-1} P_{hhv}^{21} \left(P^1 \right)^{-1}$ and $\mathcal{F}_2 = F_{hv}^2 \left(P_{hhv}^{22} \right)^{-1}$.

Proof: Applying property B.3 to (3.111) with $\mathcal{A}^T = \overline{A}_{hv}$, $\mathcal{L} = H_{hhv}^T$, $\mathcal{R} = R_{hhv}$, $\mathcal{P} = \overline{P}_{hhv}$, and $\mathcal{Q} = \overline{B}_h \overline{F}_{hv} + (*)$ gives:

$$\begin{bmatrix} \overline{A}_{hv} H_{hhv} + \overline{B}_h \overline{F}_{hv} + (*) & \overline{P}_{hhv}^T - H_{hhv}^T + \overline{A}_{hv} R_{hhv} \\ \overline{P}_{hhv} - H_{hhv} + R_{hhv}^T \overline{A}_{hv}^T & -R_{hhv} - R_{hhv}^T \end{bmatrix} < 0 . \tag{3.114}$$

Recalling the definitions of $\bar{P}_{hhv}$, $\bar{A}_{hv}$, $\bar{B}_h$, and $\bar{F}_{hv}$ with $H_{hhv} = \begin{bmatrix} H_{hv}^{11} & H_{hv}^{12} \\ H_{hh}^{21} & H_{hh}^{22} \end{bmatrix}$, $R_{hhv} = \begin{bmatrix} R_{hv}^{11} & R_{hv}^{12} \\ R_{hh}^{21} & R_{hh}^{22} \end{bmatrix}$, and after some rearrangements, (3.114) can be rewritten as:

$$\left[\begin{array}{cc|cc} \Gamma_{hh}^{11} & (*) & \multicolumn{2}{c}{\multirow{2}{*}{$(*)$}} \\ \Gamma_{hhv}^{21} & \Gamma_{hhv}^{22} & & \\ \hline \Gamma_{hhv}^{31} & \Gamma_{hhv}^{32} & \Gamma_{hv}^{33} & (*) \\ \Gamma_{hhv}^{41} & \Gamma_{hhv}^{42} & \Gamma_{hhv}^{43} & \Gamma_{hh}^{44} \end{array}\right] < 0, \tag{3.115}$$

with $\Gamma_{hh}^{11} = H_{hh}^{21} + \left(H_{hh}^{21}\right)^T$, $\Gamma_{hhv}^{21} = A_h H_{hv}^{11} + B_h F_{hv}^{1} - E_v H_{hh}^{21} + \left(H_{hv}^{22}\right)^T$, $\Gamma_{hhv}^{31} = P^1 - H_{hv}^{11} + \left(R_{hh}^{21}\right)^T$, $\Gamma_{hhv}^{22} = A_h H_{hv}^{12} + B_h F_{hv}^{2} - E_v H_{hh}^{22} + (*)$, $\Gamma_{hhv}^{32} = -H_{hv}^{12} + \left(A_h R_{hv}^{11} - E_v R_{hh}^{21}\right)^T$, $\Gamma_{hv}^{33} = -R_{hv}^{11} - \left(R_{hv}^{11}\right)^T$, $\Gamma_{hhv}^{41} = P_{hhv}^{21} - H_{hh}^{21} + \left(R_{hh}^{22}\right)^T$, $\Gamma_{hhv}^{42} = P_{hhv}^{22} - H_{hh}^{22} + \left(A_h R_{hv}^{12} - E_v R_{hh}^{22}\right)^T$, $\Gamma_{hhv}^{43} = -R_{hh}^{21} - \left(R_{hv}^{12}\right)^T$, and $\Gamma_{hh}^{44} = -R_{hh}^{22} - \left(R_{hh}^{22}\right)^T$.

Applying the relaxation lemma C.6 to (3.115) leads to conditions (3.112). For the control law expression (3.113), the state equation of the TS descriptor model (3.103) with $w = 0$ can be rewritten as:

$$\dot{x} = E_v^{-1}\left(A_h x + B_h u\right). \tag{3.116}$$

After substitution of the previous equation in (3.107), it yields:

$$u = \mathcal{F}_1 x + \mathcal{F}_2 E_v^{-1}\left(A_h x + B_h u\right), \tag{3.117}$$

which is equivalent to:

$$u = \left(I - \mathcal{F}_2 E_v^{-1} B_h\right)^{-1}\left(\mathcal{F}_1 + \mathcal{F}_2 E_v^{-1} A_h\right)x, \tag{3.118}$$

with $\mathcal{F}_1$ and $\mathcal{F}_2$ defined as in Remark 3.18. The proof is complete. ◈

Of course, if there exist uncertainties in the model or external disturbances, it is not possible to derive (3.118) and then (3.106) will include the derivative of the states. In this cases, a solution could be to use an observer which estimates $\dot{x}$.

Corollary 3.4: The solution set of (3.112) always include that of the following conditions found in Theorem 1 of (Guerra et al., 2007):

$$\bar{A}_{hv}\bar{P}_{hhv}+\bar{B}_h\bar{F}_{hv}+(*)<0. \tag{3.119}$$

Proof: The solution set of (3.112) guarantees (3.115), so that (3.114) holds. Pre- and post-multiplying (3.114) by $\begin{bmatrix} I & \bar{A}_{hv} \end{bmatrix}$ and its transpose, respectively, produce conditions in (3.119) from (Guerra et al., 2007).◆

Remark 3.19: The parameter dependency of some recent results as in (Estrada-Manzo et al., 2013) is eliminated with the previous result. Moreover, conditions in Theorem 3.15 are less restrictive than former approaches.

Example 3.9: Consider the following TS descriptor model:

$$\begin{aligned}\sum_{k=1}^{r_e} v_k(z)E_k\dot{x} &= \sum_{i=1}^{r} h_i(z)\left(A_i x+B_i u+D_i w\right)\\ y &= \sum_{i=1}^{r} h_i(z)\left(C_i x+G_i w\right),\end{aligned} \tag{3.120}$$

where $A_1=\begin{bmatrix}-4.3 & 4.8\\ -1.7 & 1\end{bmatrix}$, $A_2=\begin{bmatrix}-6 & -4.6\\ 3.9 & -1.9\end{bmatrix}$, $B_1=\begin{bmatrix}5.6\\ 0.9\end{bmatrix}$, $B_2=\begin{bmatrix}8.1\\ -3-0.01a\end{bmatrix}$, $C_1=\begin{bmatrix}1\\ 0\end{bmatrix}^T$, $C_2=\begin{bmatrix}0\\ 1\end{bmatrix}^T$, $E_1=\begin{bmatrix}0.2+b & 0\\ 0.21 & 0.03\end{bmatrix}$, $E_2=\begin{bmatrix}0.8 & 0.7\\ 0.5 & 0.68\end{bmatrix}$, $D_1=D_2=\begin{bmatrix}0.3(1-\alpha)\\ \dfrac{0.02(1-\alpha)}{5+\alpha}\end{bmatrix}$, $G_1=G_2=0$, $r=r_e=2$, $|x_1|\le 2$, $|x_2|\le 2$, $h_1=\dfrac{4-x_1^2}{4}$, $h_2=1-h_1$, $v_1=\dfrac{2-x_2}{4}$, $v_2=1-v_1$, with $a\in[-3,10]$, $b\in[-1,1.5]$, and $\alpha\in[-4,0]$. In this example, no disturbances are considered, i.e., $w=0$. When convenient, a logarithmically spaced family of values $\varepsilon\in\left\{10^{-6},10^{-5},\ldots,10^{6}\right\}$ is used to conduct some comparisons (Theorem 1 and 2 in (Estrada-Manzo et al., 2013)).

Theorem 3.15, Theorem 1 in (Guerra et al., 2007) as well as Theorem 1 in (Estrada-Manzo et al., 2013) were compared for several values of a and b. Figure 3.15 shows that all the solutions from (Guerra et al., 2007) and (Estrada-Manzo et al., 2013) are included in those of (3.112). Moreover, the quality of the solutions hereby provided is better than those of (Estrada-Manzo et al., 2013), since the latter approach requires a heuristic search of feasible solutions using a logarithmically spaced family of values of ε for each particular system under examination.

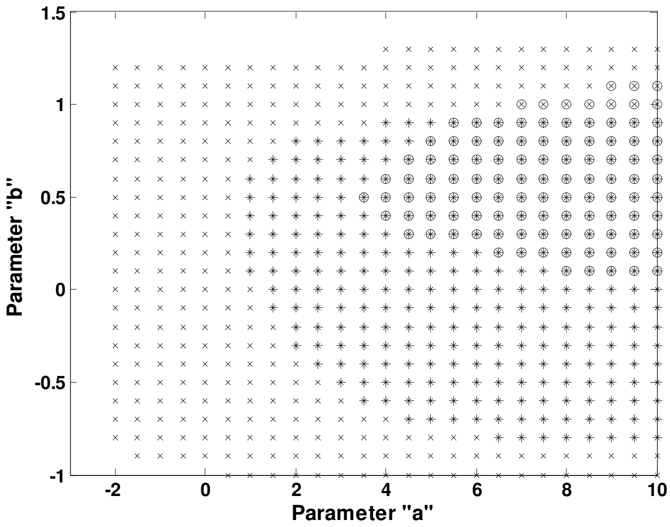


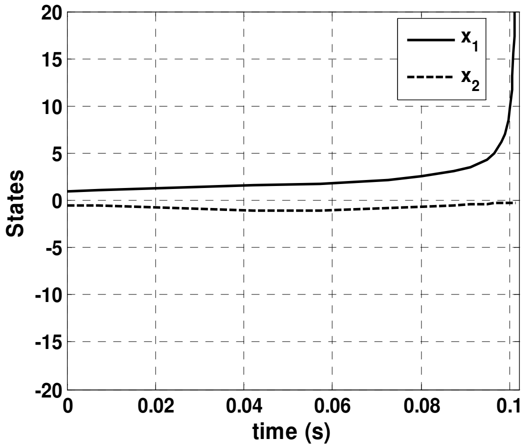
Figure 3.15. Stabilization: "$\times$" for (3.112), "$+$" for (Estrada-Manzo et al., 2013), and "$\circ$" for (Guerra et al., 2007).

Conditions (3.112) are able to find a controller for cases where the previous approaches in (Estrada-Manzo et al., 2013) and (Guerra et al., 2007) cannot: for instance, when $a = 5$ and $b = -1$, Theorem 3.15 finds a stabilizing controller of the form (3.106) with the following controller gains and common part of the Lyapunov matrix:

$$P^1 = \begin{bmatrix} 1.2767 & 0.7791 \\ 0.7791 & 1.0361 \end{bmatrix}, \quad F_{11}^1 = \begin{bmatrix} 0.4097 \\ -3.5335 \end{bmatrix}^T, \quad F_{12}^1 = \begin{bmatrix} 0.2766 \\ -3.4467 \end{bmatrix}^T, \quad F_{21}^1 = \begin{bmatrix} 1.4535 \\ 0.8865 \end{bmatrix}^T,$$

$$F_{22}^1 = \begin{bmatrix} 1.3763 \\ 0.7771 \end{bmatrix}^T, \; F_{11}^2 = \begin{bmatrix} -0.1105 \\ -0.2299 \end{bmatrix}^T, \; F_{12}^2 = \begin{bmatrix} 0.9429 \\ 1.4212 \end{bmatrix}^T, \; F_{21}^2 = \begin{bmatrix} -1.9403 \\ 0.7040 \end{bmatrix}^T, \; F_{22}^2 = \begin{bmatrix} -1.3888 \\ 2.3710 \end{bmatrix}^T.$$

Figure 3.16 shows the time response of the states for: a) open-loop ($u = 0$); b) under the controller developed via conditions (3.112). Note that in open-loop the system is unstable (left) while states converge to zero under the designed controller (right). The simulations were carried out from initial condition $x(0) = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T$. ♦



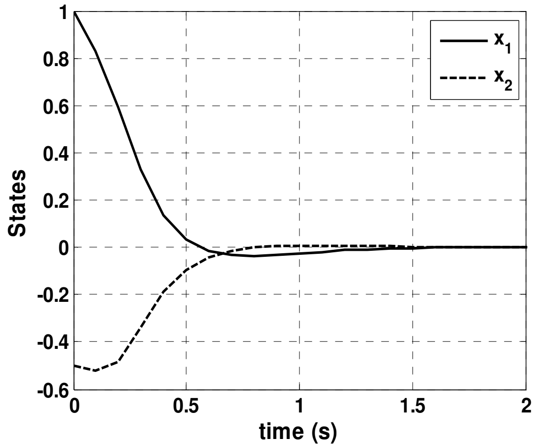


Figure 3.16. Non-perturbed time evolution of the states: a) open loop (left); b) closed-loop (right).

*$H\infty$ **Performance***

The condition for $H\infty$ attenuation criterion of a TS descriptor model (3.103) is the same as for an ordinary TS model (Boyd et al., 1994), i.e.:

$$\dot{V}(x)+\gamma^{-1}y(t)^T y(t)-\gamma w(t)^T w(t)\leq 0. \tag{3.121}$$

Recalling that $\dot{V}(\bar{x})<0$ and taking into account (3.108), (3.121) takes the following form:

$$\bar{x}^T\bar{P}_{hhv}^{-T}\left(\left(\bar{A}_{hv}+\bar{B}_h\bar{F}_{hv}\bar{P}_{hhv}^{-1}\right)\bar{x}+\bar{D}_h w\right)+(*)+\gamma^{-1}\left(\bar{C}_h\bar{x}+G_h w\right)^T\left(\bar{C}_h\bar{x}+G_h w\right)-\gamma w^T w<0,$$

which can be rearranged as:

$$\begin{bmatrix}\bar{x}\\ w\end{bmatrix}^T\begin{bmatrix}\bar{P}_{hhv}^{-T}\left(\bar{A}_{hv}+\bar{B}_h\bar{F}_{hv}\bar{P}_{hhv}^{-1}\right)+(*)+\gamma^{-1}\begin{bmatrix}\bar{C}_h^T\\ G_h^T\end{bmatrix}\begin{bmatrix}\bar{C}_h & G_h\end{bmatrix} & (*)\\ \bar{D}_h^T\bar{P}_{hhv}^{-1} & -\gamma I\end{bmatrix}\begin{bmatrix}\bar{x}\\ w\end{bmatrix}<0.$$

Applying the Schur complement to the block matrix in the middle of the previous inequality as well as congruence property with $diag\left\{P_{hhv}^T,I,I\right\}$, the previous condition is expressed as:

$$\begin{bmatrix}\bar{A}_{hv}\bar{P}_{hhv}+\bar{B}_h\bar{F}_{hv}+(*) & (*) & (*)\\ \bar{D}_h^T & -\gamma I & (*)\\ \bar{C}_h\bar{P}_{hhv} & G_h & -\gamma I\end{bmatrix}<0, \tag{3.122}$$

which leads to the following result.

Theorem 3.16: The TS descriptor model (3.103) under the control law (3.113) is asymptotically stable with disturbance attenuation γ if there exist a symmetric and positive definite matrix P^1 and matrices P_{ijk}^{21}, P_{ijk}^{22}, F_{jk}^{1}, F_{jk}^{2}, H_{jk}^{11}, H_{jk}^{12}, H_{ij}^{21}, H_{ij}^{22}, R_{jk}^{11}, R_{jk}^{12}, R_{ij}^{21}, R_{ij}^{22}, $i, j \in \{1,2,\ldots,r\}$, $k \in \{1,2,\ldots,r_e\}$, such that the following LMI conditions hold:

$$\begin{aligned} &\Upsilon_{ii}^{k} < 0,\ \forall i \in \{1,2,\cdots,r\},\ \forall k \in \{1,2,\cdots,r_e\} \\ &\frac{2}{r-1}\Upsilon_{ii}^{k} + \Upsilon_{ij}^{k} + \Upsilon_{ji}^{k} < 0,\ \forall (i,j) \in \{1,2,\cdots,r\}^2,\ i \neq j,\ \forall k \in \{1,2,\cdots,r_e\}, \end{aligned} \tag{3.123}$$

with $\Upsilon_{ij}^{k} = \begin{bmatrix} \Gamma_{ij}^{11} & (*) & (*) & (*) & (*) & (*) \\ \Gamma_{ijk}^{21} & \Gamma_{ijk}^{22} & (*) & (*) & (*) & (*) \\ \Gamma_{ijk}^{31} & \Gamma_{ijk}^{32} & \Gamma_{jk}^{33} & (*) & (*) & (*) \\ \Gamma_{ijk}^{41} & \Gamma_{ijk}^{42} & \Gamma_{ijk}^{43} & \Gamma_{ij}^{44} & (*) & (*) \\ 0 & D_i^T & 0 & 0 & -\gamma I & (*) \\ C_i P^1 & 0 & 0 & 0 & G_h & -\gamma I \end{bmatrix}$, where Γ_{ij}^{11}, Γ_{ijk}^{21}, Γ_{ijk}^{22}, Γ_{ijk}^{31}, Γ_{ijk}^{32}, Γ_{jk}^{33}, Γ_{ijk}^{41}, Γ_{ijk}^{42}, Γ_{ijk}^{43}, and Γ_{ij}^{44} are defined as in Theorem 3.15.

Proof: Property B.3 can be applied to the block entry $(1,1)$ in (3.122) with $\mathcal{A} = \bar{A}_{hv}^{T}$, $\mathcal{L}^T = H_{hhv}$, $\mathcal{R} = R_{hhv}$, $\mathcal{P} = \bar{P}_{hhv}$, and $\mathcal{Q} = \bar{B}_h \bar{F}_{hv} + (*)$, as to obtain the following inequality:

$$\begin{bmatrix} \bar{A}_{hv} H_{hhv} + \bar{B}_h \bar{F}_{hv} + (*) & (*) & (*) & (*) \\ \bar{P}_{hhv} - H_{hhv} + R_{hhv}^{T} \bar{A}_{h}^{T} & -R_{hhv} - R_{hhv}^{T} & (*) & (*) \\ \bar{D}_h^T & 0 & -\gamma I & (*) \\ \bar{C}_h \bar{P}_{hhv} & 0 & G_h & -\gamma I \end{bmatrix} < 0.$$

Following similar steps as those in Theorem 3.15, i.e.: 1) substitution of $\bar{P}_{hhv}$, $\bar{A}_{hv}$, $\bar{B}_h$, $\bar{C}_h$, $\bar{D}_h$, $\bar{F}_{hv}$, H_{hhv}, and R_{hhv}, and 2) application of lemma C.6, the previous inequality yields conditions in (3.123), thus concluding the proof. ◈

Remark 3.20: As in Corollary 3.4, it is possible to demonstrate (following the same path) that Theorem 2 of (Guerra et al., 2007) is a particular case of Theorem 3.16.

Example 3.9 (continued): Consider the TS descriptor model in (3.120) with $a = 7$, $b = 0.5$, and $w \neq 0$. The performance bound γ obtained by the theorems labelled as Theorem 2 in (Guerra et al., 2007) and (Estrada-Manzo et al., 2013), and Theorem 3.16, for different values of α, is provided in Table 3.13.

Table 3.13 shows that the performance of Theorem 3.16 is clearly better than results in (Estrada-Manzo et al., 2013) and (Guerra et al., 2007). Also, note that conditions (3.123) do not need any parameter ε to be given as in (Estrada-Manzo et al., 2013), yet the new results perform better.

Table 3.13. Comparison of $H\infty$ Performances

Approach	$\alpha=-4$	$\alpha=-3$	$\alpha=-2$	$\alpha=-1$	$\alpha=0$
Theorem 2 in (Guerra et al., 2007)	31.52	20.32	14.03	8.96	4.37
Theorem 2 in (Estrada-Manzo et al., 2013)	18.25	11.94	8.36	5.40	2.66
Theorem. 3.16	13.52	8.64	5.94	3.78	1.84

Figure 3.17 is presented in order to illustrate the behavior of the parameter γ with respect to an increasing parameter α in conditions (3.123), Theorem 2 in (Estrada-Manzo et al., 2013), and Theorem 2 in (Guerra et al., 2007), respectively. The minimal value for γ is calculated for $\alpha \in [-4,0]$.

It is possible to observe from Figure 3.17 that if parameter α increases, the minimal value of γ decreases altogether with the fact that conditions in Theorem 3.16 give better results than former approaches. ♦

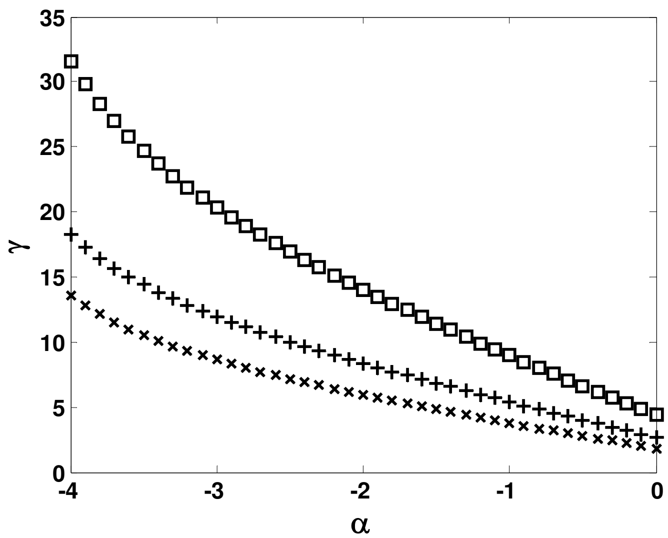


Figure 3.17. γ values: "□" for (Guerra et al., 2007), "+" for (Estrada-Manzo et al., 2013), and "×" for Theorem 3.16.

3.3.3. Stabilization via line-integral Lyapunov function

The results to be developed hereafter are based on a line-integral Lyapunov function which is an extension for TS descriptor models of that in Section 3.2.4.

Let us consider the following line-integral Lyapunov function candidate (Rhee and Won, 2006):

$$V(x) = 2\int_{\Gamma(0,x)} \mathfrak{F}(\psi)\, d\psi \,, \tag{3.124}$$

which satisfies the path-independency condition if $\mathfrak{F}(x)$ has the next structure:

$$\mathfrak{F}(x) = \left(\sum_{i=1}^{r} h_i(x)\left(\bar{P} + \mathfrak{D}_i\right)\right) x = P(x)\, x \,, \tag{3.125}$$

with

$$\bar{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n_x} \\ p_{12} & 0 & \cdots & p_{2n_x} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n_x} & p_{2n_x} & \cdots & 0 \end{bmatrix}, \quad \mathfrak{D}_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n_x n_x}^{\alpha_{in_x}} \end{bmatrix}, \quad h_i(x) = \prod_{j=1}^{n_x} \omega_j^{\alpha_{ij}}(x_j) \quad \text{where}$$

$\omega_j^{\alpha_{ij}}(x_j)$ are the WFs, and $\bar{P} + \mathfrak{D}_i = \left(\bar{P} + \mathfrak{D}_i\right)^T > 0$.

The following lemma shows that the Lyapunov function candidate satisfies the path-independency criterion (Lemma 2.1):

Lemma 3.2: The function $\mathfrak{F}(\bar{x})$ satisfies path-independent conditions if it has the next structure:

$$\mathfrak{F}(\bar{x}) = \bar{E}^T \bar{P}_h \bar{x} \,;\; \bar{E}^T \bar{P}_h = \bar{P}_h^T \bar{E} \,, \tag{3.126}$$

where $\bar{P}_h = \begin{bmatrix} P_h^1 & 0 \\ P_h^{21} & P_h^{22} \end{bmatrix}$, $P_h^1 = \left(\sum_{i=1}^{r} h_i(x)\left(\bar{P} + \mathfrak{D}_i\right)\right) x$ is defined as in (3.125), $P_h^{21} = \sum_{i=1}^{r} h_i(x) P_i^{21}$, $P_h^{22} = \sum_{i=1}^{r} h_i(x) P_i^{22}$ with P_i^{21} and P_i^{22} as free matrices.

Proof: From (3.126), the following expression

$$\mathfrak{F}(\bar{x}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_h^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} P_h^1 x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{F}_1(x) \\ \mathfrak{F}_2(\bar{x}) \end{bmatrix}, \tag{3.127}$$

leads to $\mathfrak{F}_1(x) = P_h^1 x$ (this has the same structure than (3.125)) and $\mathfrak{F}_2(\bar{x}) = 0$ (this plays no role). In order to satisfy path-independency for function (3.126), it is necessary that $\mathfrak{F}_1(x)$ holds conditions in Lemma 2.1. Due to $\mathfrak{F}_1(x) \Leftrightarrow$ (3.125), it satisfies path-independency condition.

Stability

Consider $u = 0$, then the TS descriptor model (3.104) with $w = 0$ yields

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x}\,. \tag{3.128}$$

The time-derivative of the Lyapunov function candidate in (3.124) is:

$$\dot{V}(\bar{x}) = L_g V(\bar{x}) = \mathfrak{F}^T(\bar{x})\,g(\bar{x}) + g^T(\bar{x})\,\mathfrak{F}(\bar{x}), \tag{3.129}$$

where $g(\bar{x}) = \dot{\bar{x}}$. Using (3.126) and (3.128), $\dot{V}(\bar{x}) < 0$ is satisfied if the next inequality hold:

$$\bar{P}_h^T \bar{A}_{hv} + \bar{A}_{hv}^T \bar{P}_h < 0\,. \tag{3.130}$$

Recalling the definitions of P_h and $\bar{A}_{hv}$, the previous expression is equivalent to:

$$\begin{bmatrix} A_h^T P_h^{21} + (*) & (*) \\ P_h^1 - E_v^T P_h^{21} + \left(P_h^{22}\right)^T A_h & -E_v^T P_h^{22} - \left(P_h^{22}\right)^T E_v \end{bmatrix} < 0\,. \tag{3.131}$$

Theorem 3.17: The TS descriptor model (3.103) with MF's $h_i(x)$ as in (3.125), $w = 0$ and $u = 0$ is asymptotically stable if there exists matrices $P_j^1 = \left(P_j^1\right)^T > 0$, P_j^{21}, and P_j^{22}, $j \in \{1, 2, \ldots, r\}$, such that the following LMI conditions are satisfied:

$$\begin{aligned} &\Upsilon_{ii}^k < 0, \ \forall i \in \{1, 2, \cdots, r\}, \ \forall k \in \{1, 2, \cdots, r_e\} \\ &\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \ \forall (i, j) \in \{1, 2, \cdots, r\}^2, \ i \neq j, \ \forall k \in \{1, 2, \cdots, r_e\}, \end{aligned} \tag{3.132}$$

with $\Upsilon_{ij}^k = \begin{bmatrix} A_i^T P_j^{21} + \left(P_j^{21}\right)^T A_i & (*) \\ P_j^1 - E_k^T P_j^{21} + \left(P_j^{22}\right)^T A_i & -E_k^T P_j^{22} - \left(P_j^{22}\right)^T E_k \end{bmatrix}$.

Proof: After lemma C.6 is applied to inequality (3.131), conditions (3.132) are obtained, thus concluding the proof. ◈

Remark 3.21: If $P_h^1 = P^1$, $P_h^{21} = P^{21}$, $P_h^{22} = P^{22}$, then LMI conditions in (3.132) reduce to those in Theorem 1 of (Taniguchi et al., 2000): the latter is a particular case of theorem 3.17.

Example 3.10: Consider the following TS descriptor model:

$$\sum_{k=1}^{r_e} v_k(z) E_k \dot{x} = \sum_{i=1}^{r} h_i(z) A_i x, \tag{3.133}$$

with model matrices $A_1 = \begin{bmatrix} -4.3 & 4.8 \\ -1.7 & a \end{bmatrix}$, $A_2 = \begin{bmatrix} b & -4.6 \\ -1.9 & -3.9 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0.8 & -0.5 \\ 0.21 & 1.3 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.8 & 0.7 \\ 0.5 & 0.68 \end{bmatrix}$; number of rules $r = r_e = 2$; MFs: $h_1 = \frac{x_1^2}{4}$, $h_2 = 1 - h_1$, $v_1 = \frac{x_2^2}{4}$, $v_2 = 1 - v_1$; and parameters $a \in [-100, 5]$ and $b \in [-120, 0]$.

Figure 3.18 highlight the fact that solutions of (Taniguchi et al., 2000) are all included in those of (3.132).

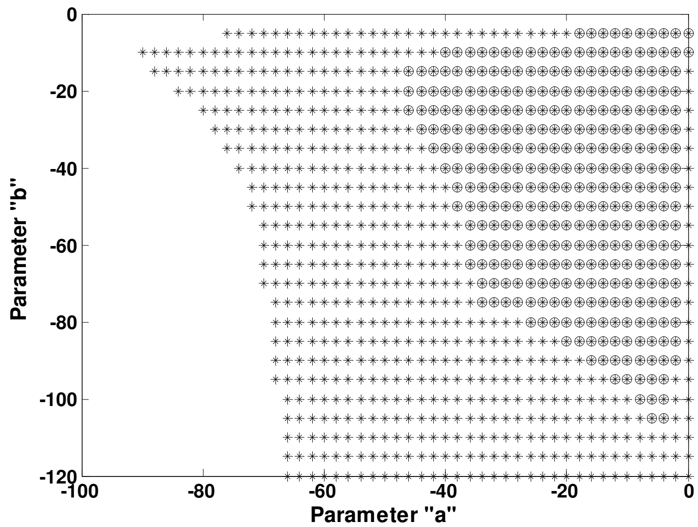


Figure 3.18. Stability: "$*$" for (3.132) and "o" for condition in (Taniguchi et al., 2000).

Now, selecting $a = -5$ and $b = -25$, a Lyapunov function of the form (3.59) can be found via conditions (3.132). The Lyapunov matrices P_i^1, $i \in \{1, 2\}$, are:

$$P_1^1 = \begin{bmatrix} 9.0426 & 5.8869 \\ 5.8869 & 9.0211 \end{bmatrix} \text{ and } P_2^1 = \begin{bmatrix} 4.2679 & 5.8869 \\ 5.8869 & 9.0211 \end{bmatrix}.$$

Figure 3.19 plots the state trajectories from four initial conditions: as expected, they all converge towards the origin. ♦

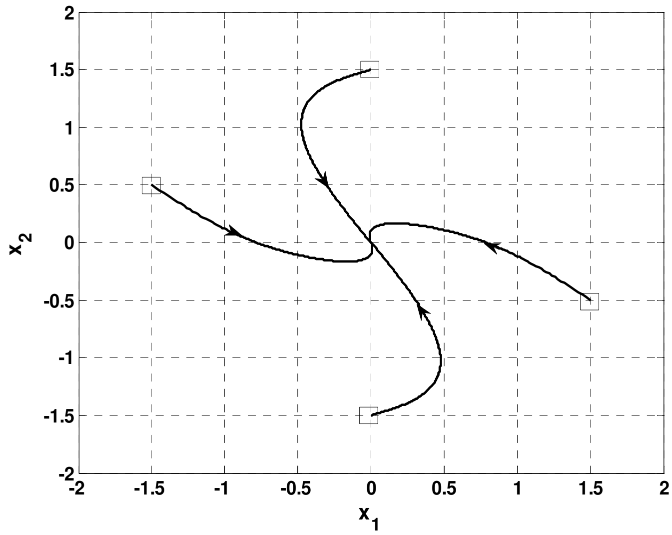


Figure 3.19. State-trajectories for $a=-5$ and $b=-25$.

Stabilization: 2nd order case

Consider the following control law (Estrada-Manzo et al., 2013):

$$u=\begin{bmatrix}F_{hv} & 0\end{bmatrix}Y_{hhv}^{-1}\bar{x}=\bar{F}_{hv}Y_{hhv}^{-1}\bar{x}, \tag{3.134}$$

where $Y_{hhv}=\begin{bmatrix}Y_{hv}^{11} & Y_{hv}^{12}\\ Y_{hh}^{21} & Y_{hh}^{22}\end{bmatrix}$, $F_{hv}=\sum_{i=1}^{r}\sum_{k=1}^{r_e}h_iv_kF_{ik}$, $Y_{hhv}=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e}h_ih_jv_kY_{ijk}$, and F_{ik}, Y_{ik}^{11}, Y_{ik}^{12}, Y_{ij}^{21}, Y_{ij}^{22}, $i,j\in\{1,2,\ldots,r\}$, $k\in\{1,2,\ldots,r_e\}$ are matrices of adequate size.

Substituting (3.134) in (3.104) with $w=0$ and properly grouping terms, the following closed-loop equality constraint is obtained:

$$\begin{bmatrix}\bar{A}_{hv}+\bar{B}_h\bar{F}_{hv}Y_{hhv}^{-1} & -I\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{E}\dot{\bar{x}}\end{bmatrix}=0. \tag{3.135}$$

Now, consider the following path-independent function

$$\mathfrak{F}(\bar{x})=\bar{E}^T\bar{P}_{hhv}^{-1}\bar{x};\ \bar{E}^T\bar{P}_{hhv}^{-1}=\bar{P}_{hhv}^{-T}\bar{E}, \tag{3.136}$$

where $\bar{P}_{hhv}=\begin{bmatrix}P_h^1 & 0\\ P_{hhv}^{21} & P_{hhv}^{22}\end{bmatrix}$, $\bar{P}_h^{-1}=\begin{bmatrix}\left(P_h^1\right)^{-1} & 0\\ -\left(P_{hhv}^{22}\right)^{-1}P_{hhv}^{21}\left(P_h^1\right)^{-1} & \left(P_{hhv}^{22}\right)^{-1}\end{bmatrix}$, $P_h^1=P(x)$ defined as in (3.125), $P_{hhv}^{21}=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e}h_ih_jv_kP_{ijk}^{21}$, $P_{hhv}^{22}=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e}h_ih_jv_kP_{ijk}^{22}$, with P_{hhv}^{22} as a regular matrix.

As shown previously and in (Márquez et al., 2013b), the Lyapunov function is path independent if and only if

$$\left(P_h^1\right)^{-1} = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}, \tag{3.137}$$

where $d_{11}^{\alpha_{i1}} = \omega_1 \bar{d}_1 + \left(1-\omega_1\right)\underline{d}_1$ and $d_{22}^{\alpha_{i2}} = \omega_2 \bar{d}_2 + \left(1-\omega_2\right)\underline{d}_2$, with $\bar{d}_1$, $\underline{d}_1$, $\bar{d}_2$, $\underline{d}_2$, and q being constants. Thus, the following inverse can be directly obtained:

$$P_h^1 = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}^{-1} = \frac{1}{d_{11}^{\alpha_{i1}} d_{22}^{\alpha_{i2}} - q^2} \begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}. \tag{3.138}$$

For convenience, expression (3.138) will be written as follows:

$$P_h^1 = \frac{X_h^1}{\left|\left(P_h^1\right)^{-1}\right|}, \tag{3.139}$$

where $X_h^1 = \begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}$ and $\left|\left(P_h^1\right)^{-1}\right| = d_{11}^{\alpha_{i1}} d_{22}^{\alpha_{i2}} - q^2$.

Theorem 3.18: A 2nd order TS descriptor model (3.103) with MF's $h_i\left(x\right)$ as in (3.125) and $w=0$ under the control law (3.134) is asymptotically stable if for a given $\varepsilon > 0$, there exist matrices $X_j^1 = \left(X_j^1\right)^T > 0$, X_{ijk}^{21}, X_{ijk}^{22}, K_{jk}, Q_{jk}^{11}, Q_{jk}^{12}, Q_{ij}^{21}, Q_{ij}^{22}, $i,j \in \{1,2,\ldots,r\}$, $k \in \{1,2,\ldots,r_e\}$, such that the following conditions hold:

$$\begin{array}{l} \Upsilon_{ii}^k < 0,\ \forall i \in \{1,2,\cdots,r\},\ \forall k \in \{1,2,\cdots,r_e\} \\ \dfrac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0,\ \forall (i,j) \in \{1,2,\cdots,r\}^2,\ i \neq j,\ \forall k \in \{1,2,\cdots,r_e\}, \end{array} \tag{3.140}$$

with $\Upsilon_{ij}^k = \begin{bmatrix} \Gamma_{ij}^{11} & (*) & (*) & (*) \\ \Gamma_{ijk}^{21} & \Gamma_{ijk}^{22} & (*) & (*) \\ \Gamma_{ijk}^{31} & \Gamma_{ijk}^{32} & \Gamma_j^{33} & (*) \\ \Gamma_{ijk}^{41} & \Gamma_{ijk}^{42} & \Gamma_{ijk}^{43} & \Gamma_{ijk}^{44} \end{bmatrix}$, $\Gamma_{ij}^{11} = Q_{ij}^{21} + \left(Q_{ij}^{21}\right)^T$, $\Gamma_{ijk}^{21} = A_i Q_{jk}^{11} - E_k Q_{ij}^{21} + B_i K_{jk} + \left(Q_{ij}^{22}\right)^T$,

$\Gamma_{ijk}^{31} = Q_{jk}^{11} - X_j^1 + \varepsilon Q_{ij}^{21}$, $\Gamma_{ijk}^{42} = Q_{ij}^{22} - X_{ijk}^{22} + \varepsilon\left(A_i Q_{jk}^{12} - E_k Q_{ij}^{22}\right)$, $\Gamma_j^{33} = -2\varepsilon X_j^1$, $\Gamma_{ijk}^{32} = Q_{jk}^{12} + \varepsilon Q_{ij}^{22}$,

$\Gamma_{ijk}^{44} = -\varepsilon X_{ijk}^{22} - \varepsilon\left(X_{ijk}^{22}\right)^T$, $\Gamma_{ijk}^{41} = Q_{ij}^{21} - X_{ijk}^{21} + \varepsilon\left(A_i Q_{jk}^{11} - E_k Q_{ij}^{21} + B_i K_{jk}\right)$, $\Gamma_{ijk}^{43} = -\varepsilon X_{ijk}^{21}$, and $\Gamma_{ijk}^{22} = A_i Q_{jk}^{12} - E_k Q_{ij}^{22} + (*)$.

Proof: The time derivative of the Lyapunov function candidate (3.124) with (3.136) can be rewritten as:

$$\begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix}^T \begin{bmatrix} 0 & \bar{P}_{hhv}^{-T} \\ \bar{P}_{hhv}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix} < 0. \tag{3.141}$$

Using Finsler's Lemma, the next inequality guarantees $\dot{V}(\bar{x}) < 0$ along the trajectories of the systems (3.103) with $w = 0$, restricted by (3.135):

$$\begin{bmatrix} \mathcal{U} \\ \mathcal{W} \end{bmatrix} \begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{F}_{hv} Y_{hhv}^{-1} & -I \end{bmatrix} + (*) + \begin{bmatrix} 0 & \bar{P}_{hhv}^{-T} \\ \bar{P}_{hhv}^{-1} & 0 \end{bmatrix} < 0. \tag{3.142}$$

Multiplying the previous expression by $\begin{bmatrix} Y_{hhv}^T & 0 \\ 0 & \bar{P}_{hhv} \end{bmatrix}$ on the left-hand side and by its transpose $\begin{bmatrix} Y_{hhv} & 0 \\ 0 & \bar{P}_{hhv}^T \end{bmatrix}$ on the right-hand side, gives

$$\begin{bmatrix} Y_{hhv}^T \mathcal{U} \\ \bar{P}_{hhv} \mathcal{W} \end{bmatrix} \begin{bmatrix} \bar{A}_{hv} Y_{hhv} + \bar{B}_h \bar{F}_{hv} & -\bar{P}_{hhv}^T \end{bmatrix} + (*) + \begin{bmatrix} 0 & Y_{hhv}^T \\ Y_{hhv} & 0 \end{bmatrix} < 0, \tag{3.143}$$

and selecting $\mathcal{U} = Y_{hhv}^{-T}$, $\mathcal{W} = \varepsilon \bar{P}_{hhv}^{-1}$, $\varepsilon > 0$, the previous expression renders:

$$\begin{bmatrix} \bar{A}_{hv} Y_{hhv} + \bar{B}_h \bar{F}_{hv} + (*) & (*) \\ Y_{hhv} - \bar{P}_{hhv} + \varepsilon\left(\bar{A}_{hv} Y_{hhv} + \bar{B}_h \bar{F}_{hv}\right) & -\varepsilon\left(\bar{P}_{hhv} + \bar{P}_{hhv}^T\right) \end{bmatrix} < 0. \tag{3.144}$$

Recalling $\bar{P}_{hhv}$ and (3.139) with $P_{hhv}^{21} = \dfrac{X_{hhv}^{21}}{\left|\left(P_h^1\right)^{-1}\right|}$, $P_{hhv}^{22} = \dfrac{X_{hhv}^{22}}{\left|\left(P_h^1\right)^{-1}\right|}$, $Y_{hv}^{11} = \dfrac{Q_{hv}^{11}}{\left|\left(P_h^1\right)^{-1}\right|}$, $Y_{hv}^{12} = \dfrac{Q_{hv}^{12}}{\left|\left(P_h^1\right)^{-1}\right|}$, $Y_{hh}^{21} = \dfrac{Q_{hh}^{21}}{\left|\left(P_h^1\right)^{-1}\right|}$, $Y_{hh}^{22} = \dfrac{Q_{hh}^{22}}{\left|\left(P_h^1\right)^{-1}\right|}$, $F_{hv} = \dfrac{K_{hv}}{\left|\left(P_h^1\right)^{-1}\right|}$, and after some operations, (3.144) can be rearranged as:

$$\frac{1}{\left|\left(P_h^1\right)^{-1}\right|} \begin{bmatrix} \Gamma_{hh}^{11} & (*) & (*) & (*) \\ \Gamma_{hhv}^{21} & \Gamma_{hhv}^{22} & (*) & (*) \\ \Gamma_{hhv}^{31} & \Gamma_{hhv}^{32} & \Gamma_h^{33} & (*) \\ \Gamma_{hhv}^{41} & \Gamma_{hhv}^{42} & \Gamma_{hhv}^{43} & \Gamma_{hhv}^{44} \end{bmatrix} < 0, \tag{3.145}$$

with $\Gamma_{hh}^{11} = Q_{hh}^{21} + \left(Q_{hh}^{21}\right)^T$, $\Gamma_{h}^{33} = -2\varepsilon X_h^1$, $\Gamma_{hhv}^{43} = -\varepsilon X_{hhv}^{21}$, $\Gamma_{hhv}^{42} = Q_{hh}^{22} - X_{hhv}^{22} + \varepsilon\left(A_h Q_{hv}^{12} - E_v Q_{hh}^{22}\right)$, $\Gamma_{hhv}^{31} = Q_{hv}^{11} - X_h^1 + \varepsilon Q_{hh}^{21}$, $\Gamma_{hhv}^{21} = A_h Q_{hv}^{11} - E_v Q_{hh}^{21} + B_h K_{hv} + \left(Q_{hh}^{22}\right)^T$, $\Gamma_{hhv}^{44} = -\varepsilon X_{hhv}^{22} - \varepsilon\left(X_{hhv}^{22}\right)^T$, $\Gamma_{hhv}^{22} = A_h Q_{hv}^{12} - E_v Q_{hh}^{22} + (*)$, $\Gamma_{hhv}^{32} = Q_{hv}^{12} + \varepsilon Q_{hh}^{22}$, $\Gamma_{hhv}^{41} = Q_{hh}^{21} - X_{hhv}^{21} + \varepsilon\left(A_h Q_{hv}^{11} - E_v Q_{hh}^{21} + B_h K_{hv}\right)$.

Due to the fact that $\dfrac{1}{\left|\left(P_h^1\right)^{-1}\right|} > 0$, (3.145) holds if:

$$\begin{bmatrix} \Gamma_{hh}^{11} & (*) & (*) & (*) \\ \Gamma_{hhv}^{21} & \Gamma_{hhv}^{22} & (*) & (*) \\ \Gamma_{hhv}^{31} & \Gamma_{hhv}^{32} & \Gamma_{h}^{33} & (*) \\ \Gamma_{hhv}^{41} & \Gamma_{hhv}^{42} & \Gamma_{hhv}^{43} & \Gamma_{hhv}^{44} \end{bmatrix} < 0. \tag{3.146}$$

Applying lemma C.6 to the previous expression gives the desired result, thus concluding the proof. ◈

Remark 3.22: If $P_h^1 = P^1$, then LMI conditions in (3.144) are the same as in Theorem 1 of (Estrada-Manzo et al., 2013), provided only second-order systems are considered.

Remark 3.23: Conditions in (3.140) are parameter-dependent LMIs; as in previous results in this thesis they are LMIs up to the choice of ε.

Example 3.11: For the sake of comparison, consider the following TS fuzzy model (Example 1 in (Estrada-Manzo et al., 2013)):

$$\sum_{k=1}^{r_e} v_k(z) E_k \dot{x} = \sum_{i=1}^{r} h_i(z)\left(A_i x + B_i u\right), \tag{3.147}$$

where $A_1 = \begin{bmatrix} -4.3 & 4.8 \\ -1.7 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a & -4.6 \\ 3.9 & -1.9 \end{bmatrix}$, $B_1 = \begin{bmatrix} 5.6 \\ 0.9 \end{bmatrix}$, $B_2 = \begin{bmatrix} 8.1 \\ b \end{bmatrix}$, $E_1 = \begin{bmatrix} 0.8+a & 0 \\ 0.21+b & 0.03 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.8 & 0.7 \\ 0.5 & 0.68 \end{bmatrix}$, $r = r_e = 2$, $h_1 = \dfrac{x_1^2}{4}$, $h_2 = 1 - h_1$, $v_1 = \dfrac{x_2^2}{4}$, $v_2 = 1 - v_1$, with $a \in [-7, 4]$ and $b \in [0.4, 2]$.

Figure 3.20 shows that all the solutions from (Estrada-Manzo et al., 2013) and Theorem 2 in (Bouarar et al., 2010) are included in those of Theorem 3.18.

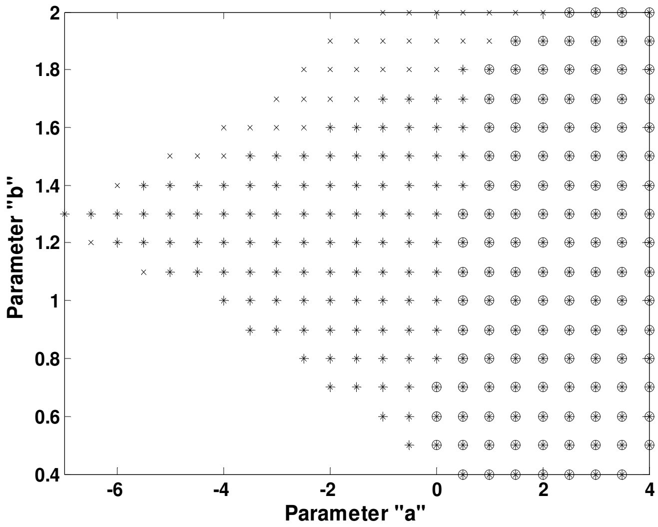


Figure 3.20. Feasibility set: "$\times$" for (3.140), "$+$" for Th. 1 in (Estrada-Manzo et al., 2013), and "o" from Th. 2 in (Bouarar et al., 2010) with $\phi_{1,2} = -1$ and $\theta_{1,2} = -1$.

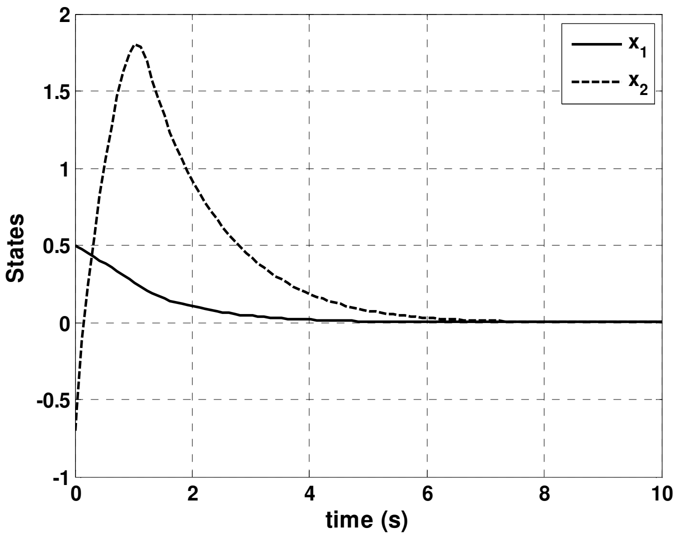


Figure 3.21. Time evolution of the states.

Unlike approaches in (Estrada-Manzo et al., 2013) and (Bouarar et al., 2010), conditions (3.140) found a controller when $a = -2$ and $b = 1.8$, with $\varepsilon = 1$; a stabilizing controller of the form (3.134) is thus obtained with the following gains and Lyapunov matrices:

$$P_1 = \begin{bmatrix} 5.5431 & -1.2761 \\ -1.2761 & 0.3003 \end{bmatrix},\ P_2 = \begin{bmatrix} 22.7101 & -1.2761 \\ -1.2761 & 0.3003 \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} -21.0092 \\ -117.3489 \end{bmatrix}^T,\ F_{12} = \begin{bmatrix} -22.3257 \\ -114.7307 \end{bmatrix}^T,\ F_{21} = \begin{bmatrix} 0.1825 \\ 0.9114 \end{bmatrix}^T, \text{ and } F_{22} = \begin{bmatrix} 0.2059 \\ 0.8113 \end{bmatrix}^T.$$

The time evolution of the states is presented in Figure 3.21: they converge to the origin. The simulation has been performed from the initial condition $x(0)=\begin{bmatrix}0.5 & -0.7\end{bmatrix}^T$. ♦

3.4. Concluding remarks

Several methodologies for state feedback controller design, both for standard as well as descriptor TS representations of continuous-time nonlinear systems, have been presented. The proposed strategies are mainly based on matrix transformations such as the Finsler's Lemma as well as a variety of Lyapunov functions such as fuzzy and line-integral. Moreover, a new Lyapunov functional has been proposed to be used instead of Lyapunov functions. Improvements on controller design via QLF and a multiple nested control law have been achieved preserving asymptotic characteristics; these improvements bring a reduction in computational burden (to help numerical solvers) as well as the inclusion of previous results (Polya's theorem) as a particular case. Moreover, these improvements have been extended using fuzzy Lyapunov functions such that a significantly reduction on conservativeness is obtained. In addition, the disturbance rejection problem has been addressed. All the strategies presented produced larger feasibility sets, preserving their LMI expression up to parameter-dependencies which can be treated via linear programming or logarithmically spaced search.

CHAPTER 4. Observer design for Takagi-Sugeno models

4.1. Introduction

This chapter provides contributions on state estimation for continuous-time nonlinear systems via TS models; they are split in two parts: the particular case where the premise vectors are based on measured variables and the general case where the premise vectors can be based on unmeasured variables.

The first part proposes progressively more relaxed observer design schemes based on: (1) a Tustin-like matrix transformation appeared in (Shaked, 2001) (2) the Finsler's Lemma (Jaadari et al., 2012), and (3) the matrix transformation in (Peaucelle et al., 2000). All of these schemes will be extended as to incorporate multiple nested convex sums (Márquez et al., 2013). Additionally, direct extensions to $H\infty$ disturbance rejection are developed.

The second part is more challenging as it faces the general case, i.e., an observer structure which facilitates handling the membership functions which depend on unmeasured variables $h_i(\hat{z})$. Former results on the subject of unmeasured variables consider the membership function error $h_i(z)-h_i(\hat{z})$ altogether with classical Lipschitz constants (Bergsten et al., 2001; Ichalal et al., 2007); this will not be the approach hereby considered. The approach in (Ichalal et al., 2011; Ichalal et al., 2012) is pursued in this thesis; the observer design is based on the differential mean value theorem. Thus, LMI conditions assuring asymptotical convergence of the state estimation error to zero are obtained; these conditions are extended to $H\infty$ performance design.

4.2. Observer design with measured premise variables: $\hat{z} = z$

This section presents some results about observer design as well as an extension to $H\infty$ disturbance rejection for continuous-time TS models, under the assumption that the premise vector depends on measured variables.

4.2.1. Problem statement

Consider the following continuous-time TS model with disturbances coupled with the state and the system output:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} h_i(z)\left(A_i x + B_i u + D_i w\right) = A_h x + B_h u + D_h w \\ y &= \sum_{i=1}^{r} h_i(z)\left(C_i x + G_i w\right) = C_h x + G_h w, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^{n_x}$ represents the system state vector, $u \in \mathbb{R}^{n_u}$ the input vector, $y \in \mathbb{R}^{n_y}$ the measured output vector, $w \in \mathbb{R}^{n_w}$ the vector of external disturbances; $h_i(\cdot)$, $i \in \{1, 2, \ldots, r\}$ are the membership functions which depend on the vector of premise variables $z \in \mathbb{R}^p$; and matrices $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $C_i \in \mathbb{R}^{n_y \times n_x}$, $D_i \in \mathbb{R}^{n_x \times n_w}$, and $G_i \in \mathbb{R}^{n_y \times n_w}$ result from modeling a given nonlinear system.

In order to estimate the states which are not available, the following observer, mimicking the TS model in (4.1), is proposed considering $\hat{z} = z$:

$$\begin{aligned} \dot{\hat{x}} &= A_h \hat{x} + B_h u + \mathcal{H}^{-1}(z)\mathcal{K}(z)\left(y - \hat{y}\right) \\ \hat{y} &= C_h \hat{x}, \end{aligned} \tag{4.2}$$

with $\hat{x} \in \mathbb{R}^{n_x}$ is the observer state, $\hat{y} \in \mathbb{R}^{n_y}$ the estimated measured output, $e = x - \hat{x}$ the estimation error, and $\mathcal{H}(z) \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{K}(z) \in \mathbb{R}^{n_x \times n_y}$ are matrix functions of the premise vector z to be designed in the sequel. To ease notation, arguments of these matrix functions will be omitted.

Remark 4.1: As it will be seen in detail, observer matrices $\mathcal{H}$ and $\mathcal{K}$ can be chosen as the corresponding approach requires, for instance, $\mathcal{H} = H_{hh\cdots h}$ and $\mathcal{K} = K_{hh\cdots h}$ for the observer gains with multiple nested summations or $\mathcal{H} = H_h$ and $\mathcal{K} = K_h$ for single sums. The link between Lyapunov function and the observer design can be removed when $\mathcal{H} = H_h$, for instance; this sort of "decoupling" implies more flexibility in the conditions to be satisfied.

The notation for "q" multiple nested convex sums, given in Table 2.1, will be used in the sequel; i.e.: $\Upsilon_{\bar{h}} = \Upsilon_{\underbrace{hh\cdots h}_{q}} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r} h_{i_1}(z)h_{i_2}(z)\cdots h_{i_q}(z)\Upsilon_{i_1 i_2\cdots i_q}$.

According to the notation above, the estimation error dynamics is described as:

$$\dot{e} = \left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)e + \left(D_h - \mathcal{H}^{-1}\mathcal{K}G_h\right)w. \tag{4.3}$$

Now, consider the following quadratic Lyapunov function (QLF) candidate with $P = P^T > 0$:

$$V(e) = e^T Pe. \tag{4.4}$$

Condition $\dot{V}(e) < 0$ is satisfied if:

$$e^T P\dot{e} + \dot{e}^T Pe < 0. \tag{4.5}$$

4.2.2. Observer design with measured premise variables

Three different approaches will be considered in the following developments to derive conditions for observer design of TS models: a first one based on a matrix transformation from (Shaked, 2001), a second one via Finsler's Lemma (de Oliveira and Skelton, 2001), and a third one based on the matrix transformation of (Peaucelle et al., 2000); all of them will incorporate the necessary adjustments such that the conservativeness of the conditions is progressively reduced through sum relaxations (Márquez et al., 2013).

To begin with, the estimation error dynamics is considered without disturbances, i.e., $w = 0$.

First approach.

Condition in (4.5) is equivalent to:

$$P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) + \left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T P < 0. \tag{4.6}$$

Inspired by (Shaked, 2001), considering a small enough $\varepsilon > 0$, it is clear that the following condition is equivalent to (4.6):

$$P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) + \left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T P + \varepsilon\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) < 0, \tag{4.7}$$

from which the next rewriting can be done multiplying by ε and adding $P - P$:

$$\varepsilon P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) + (*) + \varepsilon^2\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) + P - P < 0. \quad (4.8)$$

This expression can be rearranged as:

$$\left(I + \varepsilon\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T\right)P\left(I + \varepsilon\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)\right) - P < 0, \quad (4.9)$$

which by Schur complement will be equivalent to:

$$\begin{bmatrix} P & (*) \\ I + \varepsilon\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) & P^{-1} \end{bmatrix} > 0. \quad (4.10)$$

The previous expression can be pre-multiplied by $\begin{bmatrix} I & 0 \\ 0 & \mathcal{H} \end{bmatrix}$ and post-multiplied by $\begin{bmatrix} I & 0 \\ 0 & \mathcal{H}^T \end{bmatrix}$ to produce the equivalent condition:

$$\begin{bmatrix} P & (*) \\ \mathcal{H} + \varepsilon\left(\mathcal{H}A_h - \mathcal{K}C_h\right) & \mathcal{H}P^{-1}\mathcal{H}^T \end{bmatrix} > 0. \quad (4.11)$$

Selecting $\mathcal{H} = H_{\bar{h}}$ and $\mathcal{K} = K_{\bar{h}}$, the following theorem is stated.

Theorem 4.1. The estimation error model (4.3) with $w = 0$ is asymptotically stable if $\exists \varepsilon > 0$ and matrices $P = P^T > 0$, $H_{i_1 i_2 \cdots i_q}$, and $K_{i_1 i_2 \cdots i_q}$, $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, r\}$ of proper dimensions such that the following conditions hold:

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \rho\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall\left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1, 2, \ldots, r\}^{q+1}, \quad (4.12)$$

with $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} -P & (*) \\ H_{i_1 i_2 \cdots i_q} + \varepsilon\left(H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0}\right) & P - H_{i_1 i_2 \cdots i_q} - H_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$ and $\rho\left(i_0, i_1, i_2, \ldots, i_q\right)$ as the set of permutations with repeated elements of indexes $i_0, i_1, i_2, \ldots, i_q$.

Proof: Using the property B.7 with $\mathcal{Q}^T = \mathcal{H} = H_{\bar{h}}$ and $\mathcal{P} = P$, it is clear that $H_{\bar{h}} P^{-1} H_{\bar{h}}^T \geq H_{\bar{h}} + H_{\bar{h}}^T - P$, which allows guaranteeing (4.11) if the following holds:

$$\begin{bmatrix} -P & (*) \\ H_{\bar{h}} + \varepsilon\left(H_{\bar{h}} A_h - K_{\bar{h}} C_h\right) & P - H_{\bar{h}} - H_{\bar{h}}^T \end{bmatrix} < 0. \quad (4.13)$$

Applying relaxation lemma C.4 to (4.13) leads to conditions (4.12), which concludes the proof. ◈

Second approach.

Condition in (4.5) can be rearranged as:

$$\dot{V} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} < 0, \tag{4.14}$$

altogether with the following equality constraint:

$$\begin{bmatrix} A_h - \mathcal{H}^{-1}\mathcal{K}C_h & -I \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = 0, \tag{4.15}$$

arising from (4.3) with $w = 0$. If inequality (4.14) under equality constraint (4.15) holds, it is equivalent, by Finsler's Lemma, to the following:

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{V} \\ \mathcal{W} \end{bmatrix} \begin{bmatrix} A_h - \mathcal{H}^{-1}\mathcal{K}C_h & -I \end{bmatrix} + (*) < 0. \tag{4.16}$$

In order to get LMI conditions and recover the “classical” quadratic case, let $\mathcal{V} = \mathcal{H}$ and $\mathcal{W} = \varepsilon\mathcal{H}$ with $\varepsilon > 0$, then (4.16) holds if:

$$\begin{bmatrix} \mathcal{H}A_h - \mathcal{K}C_h + \left(\mathcal{H}A_h - \mathcal{K}C_h\right)^T & (*) \\ P - \mathcal{H}^T + \varepsilon\left(\mathcal{H}A_h - \mathcal{K}C_h\right) & -\varepsilon\left(\mathcal{H} + \mathcal{H}^T\right) \end{bmatrix} < 0, \tag{4.17}$$

and choosing $\mathcal{H} = H_{\bar{h}}$ and $\mathcal{K} = K_{\bar{h}}$, the following theorem arises.

Theorem 4.2. The estimation error model (4.3) with $w = 0$ is asymptotically stable if $\exists \varepsilon > 0$ and matrices $P = P^T > 0$, $H_{i_1 i_2 \cdots i_q}$, and $K_{i_1 i_2 \cdots i_q}$, $i_1, \ldots, i_q \in \{1, \ldots, r\}$ of proper dimensions such that the next conditions hold:

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \wp\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall\left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1, 2, \ldots, r\}^{q+1}, \tag{4.18}$$

with $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0} + \left(H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0}\right)^T & (*) \\ P - H^T_{i_1 i_2 \cdots i_q} + \varepsilon\left(H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0}\right) & -\varepsilon\left(H_{i_1 i_2 \cdots i_q} + H^T_{i_1 i_2 \cdots i_q}\right) \end{bmatrix}.$

Proof: After substitution of $\mathcal{H} = H_{\bar{h}}$ and $\mathcal{K} = K_{\bar{h}}$ in (4.17), it yields

$$\begin{bmatrix} H_{\bar{h}}A_h - K_{\bar{h}}C_h + \left(H_{\bar{h}}A_h - K_{\bar{h}}C_h\right)^T & (*) \\ P - H_{\bar{h}}^T + \varepsilon\left(H_{\bar{h}}A_h - K_{\bar{h}}C_h\right) & -\varepsilon\left(H_{\bar{h}} + H_{\bar{h}}^T\right) \end{bmatrix} < 0. \tag{4.19}$$

Applying lemma C.4 to (4.19) gives the conditions presented in (4.18); thus producing the desired result. ◈

Third approach.

Condition in (4.5) is equivalent to:

$$PA_h - P\mathcal{H}^{-1}\mathcal{K}C_h + \left(PA_h - P\mathcal{H}^{-1}\mathcal{K}C_h\right)^T < 0, \tag{4.20}$$

which leads to the next theorem.

Theorem 4.3. The estimation error model (4.3) with $w=0$ is asymptotically stable if there exist matrices $P = P^T > 0$, $R_{i_1 i_2 \cdots i_q}$, $H_{i_1 i_2 \cdots i_q}$, and $K_{i_1 i_2 \cdots i_q}$, $i_1, i_2, \ldots, i_q \in \{1,2,\ldots,r\}$ of proper dimensions such that the following conditions hold:

$$\sum_{i_0 i_1 i_2 \cdots i_q \in \rho\left(i_0 i_1 i_2 \cdots i_q\right)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0, \quad \forall\left(i_0, i_1, i_2, \ldots, i_q\right) \in \{1,2,\ldots,r\}^{q+1}, \tag{4.21}$$

with $\Upsilon_{i_0 i_1 i_2 \cdots i_q} = \begin{bmatrix} H_{i_1 i_2 \cdots i_q}A_{i_0} - K_{i_1 i_2 \cdots i_q}C_{i_0} + \left(H_{i_1 i_2 \cdots i_q}A_{i_0} - K_{i_1 i_2 \cdots i_q}C_{i_0}\right)^T & (*) \\ P - H_{i_1 i_2 \cdots i_q}^T + R_{i_1 i_2 \cdots i_q}^T A_{i_0} & -R_{i_1 i_2 \cdots i_q} - R_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$.

Proof: Assuming $\mathcal{H} = P$, $\mathcal{K} = K_{\bar{h}}$, and applying property B.3 with $\mathcal{A} = A_h$, $\mathcal{L} = H_{\bar{h}}$, $\mathcal{R} = R_{\bar{h}}$, $\mathcal{P} = P$, $\mathcal{Q} = -K_{\bar{h}}C_h - C_h^T K_{\bar{h}}^T$ to (4.20), it writes:

$$\begin{bmatrix} H_{\bar{h}}A_h - K_{\bar{h}}C_h + \left(H_{\bar{h}}A_h - K_{\bar{h}}C_h\right)^T & (*) \\ P - H_{\bar{h}}^T + R_{\bar{h}}^T A_h & -R_{\bar{h}} - R_{\bar{h}}^T \end{bmatrix} < 0, \tag{4.22}$$

and when lemma C.4 is applied to the previous expression, conditions in (4.21) are obtained, which ends the proof. ◈

Remark 4.2: A parameter $\varepsilon > 0$ appears for the first and second approach as usual when dealing with Finsler's Lemma (Jaadari et al., 2012). Effectively, in order to include the quadratic case it is mandatory that $\mathcal{W}$ can go to 0, therefore setting $\mathcal{W} = \varepsilon\mathcal{H}$ answers to this constraint.

- For example, considering the first approach, with $H_{\bar{h}} = P$, $K_{\bar{h}} = K_h$, using the Schur complement of (4.13), and after some manipulations, we have:

$$PA_h - K_h C_h + \left(PA_h - K_h C_h\right)^T + \varepsilon\left(PA_h - K_h C_h\right)^T P^{-1}\left(PA_h - K_h C_h\right) < 0\,, \tag{4.23}$$

which proves the referred inclusion when $\varepsilon > 0$ is small enough.

- The inclusion proof for the second approach is directly obtained by the Schur complement to (4.19) with $H_{\bar{h}} = P$, $K_{\bar{h}} = K_h$, and choosing ε small enough, which can be noticed in the next conditions:

$$PA_h - K_h C_h + \left(PA_h - K_h C_h\right)^T + \frac{1}{2}\varepsilon\left(PA_h - K_h C_h\right)^T P^{-1}\left(PA_h - K_h C_h\right) < 0\,. \tag{4.24}$$

Remark 4.3: The inclusion of the quadratic case is also guaranteed in the third approach, even if it is not parameter-dependent. It is clear hat with $K_{\bar{h}} = K_h$ and pre- and post-multiplying (4.22) by $\begin{bmatrix} I & A_h \end{bmatrix}$ and its transpose, respectively, the classical conditions are recovered:

$$PA_h - K_h C_h + \left(PA_h - K_h C_h\right)^T < 0\,. \tag{4.25}$$

Remark 4.4: Results in Theorems 4.1 and 4.2 are parameter-dependent LMIs; they depend on the choice of $\varepsilon > 0$. This parameter is employed in several works concerning linear parameter varying (LPV) systems (de Oliveira and Skelton, 2001; Shaked, 2001; Oliveira et al., 2011; Jaadari et al., 2012): they are normally prefixed values belonging to a logarithmically spaced family of values, such as: $\varepsilon \in \left\{10^{-6}, 10^{-5}, \ldots, 10^{6}\right\}$, which avoids performing an exhaustive linear search.

Remark 4.5: Conditions in Theorem 4.3 are LMI, i.e., they do not require a *priori* parameters. Nevertheless, they require more slack variables. Table 4.1 presents the number of LMI conditions (N_L) and decision variables (N_D) of each approach, for comparison.

Table 4.1. Number of LMI conditions and decision variables.

Approaches	N_L	N_D
Theorem 4.1	$\binom{r+q}{q+1}+1$	$\frac{n_x}{2}\left(n_x+1\right)+n_x n_u r^q + n_x^2 r^q$
Theorem 4.2	$\binom{r+q}{q+1}+1$	$\frac{n_x}{2}\left(n_x+1\right)+n_x n_u r^q + n_x^2 r^q$
Theorem 4.3	$\binom{r+q}{q+1}+1$	$\frac{n_x}{2}\left(n_x+1\right)+n_x n_u r^q + 2n_x^2 r^q$

In Table 4.1, $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$.

Example 4.1: Consider the following TS model:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{4} h_i(z)(A_i x + D_i w) \\ y &= \sum_{i=1}^{4} h_i(z)(C_i x + G_i w), \end{aligned} \tag{4.26}$$

with $A_1 = \begin{bmatrix} -\frac{1}{2}(1-a) - \frac{4}{1+b} & -0.1 \\ 1.4 & -0.7 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & -1.3 \\ 0.2 & 0.3 \end{bmatrix}$, $A_3 = \begin{bmatrix} -0.3 & -1.1 \\ 0.9 & 0.6 \end{bmatrix}$,

$A_4 = \begin{bmatrix} -0.8 & -0.8 \\ 0.4 & 0.3 \end{bmatrix}$, $C_1 = [-0.1 \quad 0]$, $C_2 = [0.3 + 0.1b \quad 0]$, $C_3 = [0.04 \quad 0]$, $C_4 = [0.2 \quad 0]$,

$D_1 = D_3 = \left[0.3(1.2-\alpha) \quad 0.02\frac{1-\alpha}{4+\alpha}\right]^T$, $D_2 = D_4 = \left[0.2(1.2-\alpha) \quad 0.01\frac{1-\alpha}{5+\alpha}\right]^T$, $G_i = 0$,

$i \in \{1,2,3,4\}$, $x_1 \in [-2,2]$, $z_1 = \sin(x_1)$, $z_2 = x_1^2$, $\omega_0^1(z) = \frac{1-\sin(x_1)}{2}$, $\omega_1^1(z) = \frac{1+\sin(x_1)}{2}$,

$\omega_0^2(z) = \frac{4-x_1^2}{4}$, $\omega_1^1(z) = \frac{x_1^2}{4}$, $h_1(z) = \omega_0^1(z)\omega_0^2(z)$, $h_2(z) = \omega_1^1(z_1)\omega_0^2(z_2)$,

$h_3(z) = \omega_0^1(z)\omega_1^2(z)$, $h_4(z) = \omega_1^1(z)\omega_1^2(z)$ with $a \in [0,1]$. In this example no disturbances are considered, i.e., $w = 0$. Parameter $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$ has been considered when necessary.

Theorems 4.1, 4.2, 4.3, and conditions in (4.25) were compared considering different values for q ($q=1$, $q=2$, and $q=3$). In Table 4.2, the maximum value of b achieved for each approach is presented considering $a \in \{0, 0.5, 1\}$.

Table 4.2. Comparison of maximum value of b.

Approach	$q=1$			$q=2$			$q=3$		
	$a=0$	$a=0.5$	$a=1$	$a=0$	$a=0.5$	$a=1$	$a=0$	$a=0.5$	$a=1$
QS (4.25)	6.2	4.1	2.9	–	–	–	–	–	–
Theorem 4.1	6.2	4.1	2.9	6.6	4.3	3.1	7.3	4.7	3.3
Theorem 4.2	6.4	4.3	3	7.5	4.8	3.3	8.4	5.2	3.5
Theorem 4.3	8.4	5.1	3.5	11.1	6.1	3.9	12	6.2	3.9

It is clear to observe that conditions in Theorem 4.3 give feasible solutions for higher values respect to the other approaches. Moreover, it is not necessary to use a heuristic search of feasible solutions using a logarithmically spaced family of values of ε as in Theorems 4.1 and 4.2. Nevertheless, it is not possible to show that these approaches are equivalent or any inclusion between them. They remain as different proposals to solve the problem.

Now, using the conditions in Theorem 4.3 for state estimation (measured premises) and selecting the following parameters $q = 2$, $a = 0$, and $b = 10$, a feasible solution has been found; notice that it is not possible with the other approaches.

The gains for the observer and Lyapunov matrix are given by

$$P = \begin{bmatrix} 0.2175 & 0.0866 \\ 0.0866 & 0.1537 \end{bmatrix}, \quad K_{11} = \begin{bmatrix} 0.5017 \\ -0.1980 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} 2.1101 \\ -1.1060 \end{bmatrix}, \quad K_{13} = \begin{bmatrix} -0.2748 \\ 1.4931 \end{bmatrix},$$

$$K_{14} = \begin{bmatrix} 0.9728 \\ -1.0092 \end{bmatrix}, \; K_{21} = 0, \; K_{22} = \begin{bmatrix} 0.9071 \\ -0.1646 \end{bmatrix}, \; K_{23} = \begin{bmatrix} 1.94 \\ -0.3468 \end{bmatrix}, \; K_{24} = \begin{bmatrix} 0.9593 \\ -0.2867 \end{bmatrix}, \; K_{31} = 0,$$

$$K_{32} = 0, \; K_{33} = \begin{bmatrix} 2.4202 \\ -1.0851 \end{bmatrix}, \; K_{34} = \begin{bmatrix} 1.5366 \\ -0.7265 \end{bmatrix}, \; K_{41} = 0, \; K_{42} = 0, \; K_{43} = 0, \; K_{44} = \begin{bmatrix} 1.1603 \\ -0.4704 \end{bmatrix}.$$

The estimation error for a trajectory of the states is presented in Figure 4.1. The initial conditions are $\begin{bmatrix} 1.5 & -0.7 \end{bmatrix}^T$, while the estimated ones are $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$. The time evolution of the states is shown in Figure 4.2; $u = 0$ has been considered. It is possible to observe that the estimation error goes to zero despite the fact that the states remain oscillating.♦

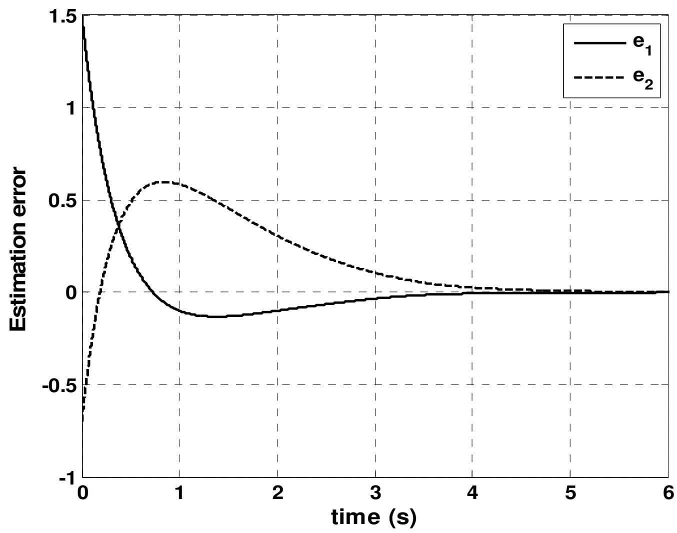


Figure 4.1. Time evolution of the estimation error.

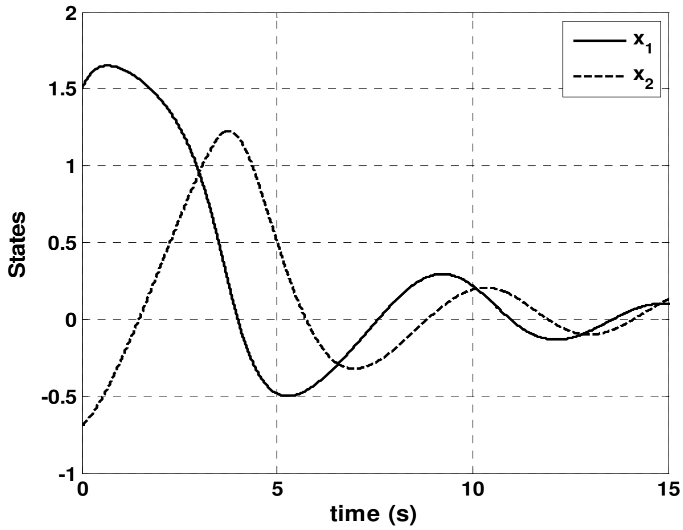


Figure 4.2. Time evolution of the states: x_1 (solid line) and x_2 (dashed line).

4.2.3. $H\infty$ disturbance rejection

In this case $w \neq 0$. The $H\infty$ attenuation criterion for the estimation error dynamics is given by

$$\sup_{\|w(t)\|_2 \neq 0} \frac{\|e(t)\|_2}{\|w(t)\|_2} \leq \gamma \,, \tag{4.27}$$

where $\gamma > 0$ is a positive scalar representing the disturbance level of attenuation (Boyd et al., 1994). This is equivalent to the existence of a Lyapunov functional candidate $V(e)$ such that

$$\dot{V}(e) + e^T e - \gamma^2 w^T w \leq 0 \,. \tag{4.28}$$

Then, using condition in (4.5), the following inequality is equivalent to (4.28):

$$e^T P\dot{e} + \dot{e}^T Pe + e^T e - \gamma^2 w^T w \leq 0 \,. \tag{4.29}$$

It can be rearranged as:

$$\begin{bmatrix} e \\ w \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} I & 0 & P \\ 0 & -\gamma^2 I & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ w \\ \dot{e} \end{bmatrix} \leq 0 \,, \tag{4.30}$$

or, substituting (4.3):

$$\begin{bmatrix} e \\ w \end{bmatrix}^T \begin{bmatrix} P\left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right) + \left(A_h - \mathcal{H}^{-1}\mathcal{K}C_h\right)^T P + I & PD_h \\ D_h^T P - G_h^T \mathcal{K}^T \mathcal{H}^{-T} P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \leq 0 \,. \tag{4.31}$$

The matrix transformations in section 4.2.2 will be used in the following developments: 1) (4.31) for the first approach with matrix transformation in (Shaked, 2001) (Theorem 4.4); 2) (4.30) and (4.3) for the second approach with Finsler's Lemma in (de Oliveira and Skelton, 2001) (Theorem 4.5); 3) (4.31) for the third approach with matrix transformation (Peaucelle et al., 2000) (Theorem 4.6). In Table 4.3 the obtained conditions for each approach are presented.

Table 4.3. Conditions of $H\infty$ observer design for new approaches.

Approach	Conditions: $\sum_{i_0 i_1 i_2 \cdots i_q \in \rho(i_0 i_1 i_2 \cdots i_q)} \Upsilon_{i_0 i_1 i_2 \cdots i_q} < 0$, $\forall (i_0, i_1, i_2, \ldots, i_q) \in \{1, 2, \ldots, r\}^{q+1}$, $\gamma = \sqrt{\bar{\gamma}}$.	Eq.
Th. 4.4 $G_h = 0$ $\mathcal{H} = H_{\bar{h}}$ $\mathcal{K} = K_{\bar{h}}$	$\begin{bmatrix} -P + \varepsilon I & (*) & (*) \\ \varepsilon D_{i_0}^T P & -\varepsilon \bar{\gamma} I & (*) \\ H_{i_1 i_2 \cdots i_q} + \varepsilon \left(H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0} \right) & 0 & P - H_{i_1 i_2 \cdots i_q} - H_{i_1 i_2 \cdots i_q}^T \end{bmatrix}$	(4.32)
Th. 4.5 $\mathcal{H} = H_{\bar{h}}$ $\mathcal{K} = K_{\bar{h}}$	$\begin{bmatrix} H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0} + (*) + I & (*) & (*) \\ D_{i_0}^T H_{i_1 i_2 \cdots i_q}^T - G_{i_0}^T K_{i_1 i_2 \cdots i_q}^T & -\bar{\gamma} I & (*) \\ P - H_{i_1 i_2 \cdots i_q}^T + \varepsilon \left(H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0} \right) & \varepsilon \left(H_{i_1 i_2 \cdots i_q} D_{i_0} - K_{i_1 i_2 \cdots i_q} G_{i_0} \right) & -\varepsilon H_{i_1 i_2 \cdots i_q} - (*) \end{bmatrix}$	(4.33)
Th. 4.6 $\mathcal{H} = P$ $\mathcal{K} = K_{\bar{h}}$	$\begin{bmatrix} H_{i_1 i_2 \cdots i_q} A_{i_0} - K_{i_1 i_2 \cdots i_q} C_{i_0} + (*) + I & (*) & (*) \\ D_{i_0}^T P - G_{i_0}^T K_{i_1 i_2 \cdots i_q}^T & -\bar{\gamma} I & (*) \\ P - H_{i_1 i_2 \cdots i_q}^T + R_{i_1 i_2 \cdots i_q}^T A_{i_0} & 0 & -R_{i_1 i_2 \cdots i_q} - R_{i_1 i_2 \cdots i_q}^T \end{bmatrix}.$	(4.34)

Example 4.1 (continued): Consider the TS model in (4.26) with $a = 1$, $b = 2.9$, and $w \neq 0$. For sake of comparison, recall also that the classical conditions are:

$$\begin{bmatrix} PA_h - K_h C_h + \left(PA_h - K_h C_h \right)^T + I & PD_h \\ D_h^T P & -\bar{\gamma} I \end{bmatrix} < 0. \tag{4.35}$$

The performance bounds γ obtained by conditions in Theorems 4.4, 4.5, 4.6, and the classical conditions (4.35), for different values of α, are provided in Table 4.4.

Table 4.4. Comparison of $H\infty$ performances.

Approach	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
Conditions (4.35)	71.5	41.7	11.9
Theorem 4.4	71.5	41.7	11.9
Theorem 4.5	27.5	16	4.6
Theorem 4.6	10.9	6.4	1.8

Table 4.4 shows that the performance achieved by Theorem 4.6 is clearly better than in other approaches.

Figures 4.3, 4,4, and 4.5 are presented in order to illustrate the behavior of the attenuation level γ with respect to parameter q in conditions (4.32), (4.33), and (4.34), respectively. The minimal value for γ is calculated for $\alpha \in [0,1]$.

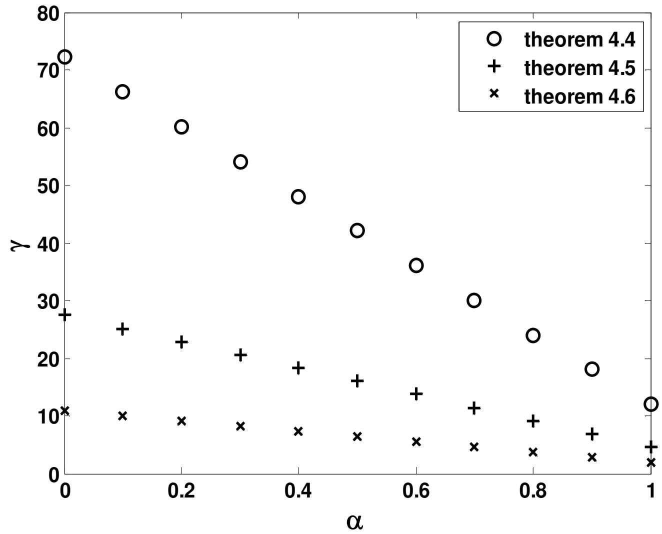


Figure 4.3. γ values for Theorems 4.4, 4.5, and 4.6 with $q = 1$.

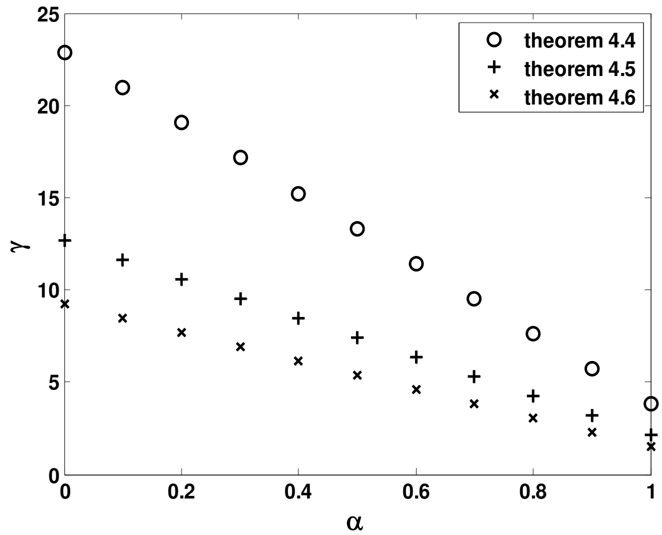


Figure 4.4. γ values for Theorems 4.4, 4.5, and 4.6 with $q = 2$.

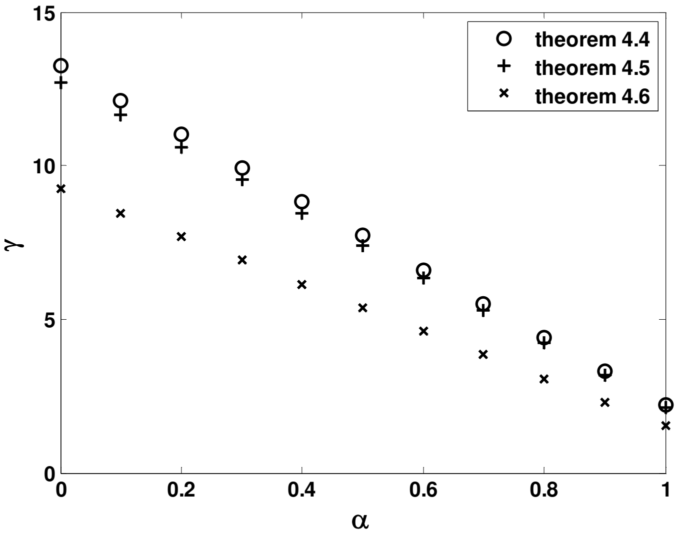


Figure 4.5. γ values for Theorems 4.4, 4.5, and 4.6 with $q = 3$.

From the figures above it is clear that if parameter q increases, the minimal value of γ decreases. Conditions in Theorem 4.6 always give better results than the others. ♦

4.3. Observer design with unmeasured premise variables: $\hat{z} \neq z$

This section provides some results concerned with state estimation for continuous-time nonlinear systems. It is shown that the differential mean value theorem and a quadratic Lyapunov function can be used to provide conditions in the form of linear matrix inequalities, guaranteeing the state estimation error to be asymptotically driven to zero. As in the previous section, an extension to $H\infty$ disturbance rejection is also presented.

4.3.1. Problem statement

Consider the continuous-time TS model with disturbances in (4.1). As the development presented hereafter concerns state variable estimations, it is convenient to separate the expressions of the functions $h_i(\cdot)$ which depend exclusively on measured premise variables (z_α) and the ones depending on non-measured premise variable (z_β). This separation leads to the next equivalent representation of (4.1):

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\beta_j(z_\beta)\left(A_{ij}x + B_{ij}u + D_{ij}w\right) = A_{\alpha\beta}x + B_{\alpha\beta}u + D_{\alpha\beta}w \\
y &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\beta_j(z_\beta)\left(C_{ij}x + G_{ij}w\right) = C_{\alpha\beta}x + G_{\alpha\beta}w,
\end{aligned} \tag{4.36}$$

where $\alpha_i\left(z_\alpha\right)$, $i\in\left\{1,\ldots,2^{p_\alpha}\right\}$ are positive functions depending on measured premise variables and $\beta_j\left(z_\beta\right)$, $j\in\left\{1,\ldots,2^{p_\beta}\right\}$, the positive functions depending on non-measured premise variable, with $r_\alpha=2^{p_\alpha}\in\mathbb{N}$ and $r_\beta=2^{p_\beta}\in\mathbb{N}$. When all the premises are measured $\alpha_i\left(z_\alpha\right)=h_i\left(z\right)$, $\beta_j\left(z_\beta\right)=1$; when none of them is measurable, $\alpha_i\left(z_\alpha\right)=1$, $\beta_j\left(z_\beta\right)=h_j\left(z\right)$. Notice also that: $\sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i\left(z_\alpha\right)\beta_j\left(z_\beta\right)=1$, $\sum_{i=1}^{r_\alpha}\alpha_i\left(z_\alpha\right)=1$, and $\sum_{i=1}^{r_\beta}\beta_j\left(z_\beta\right)=1$.

Assumptions: In the following, $\mathcal{O}_{xu}$ denotes the operating set of the TS observer in which: 1) TS model (4.1) perfectly match the non-linear model; 2) there exist known scalars λ_x and λ_u such that $\|x\|\le\lambda_x$ and $\|u\|\le\lambda_u$ hold; 3) the observer is supposed to converge to the system's state. From these bounds, scalars σ_j are deduced such that $\left\|\frac{\partial\beta_j}{\partial z_\beta}\right\|\le\sigma_j$ in $\mathcal{O}_{xu}$.

The following example shows the relation between TS models (4.1) and (4.36).

Example 4.2: Consider the following nonlinear model of the bioreactor from (Farza et al., 2014):

$$\begin{aligned}\dot{x}_1&=\frac{\mu^* x_1 x_2}{K_c x_1+x_2}-x_1 u\\ \dot{x}_2&=-\frac{k\mu^* x_1 x_2}{K_c x_1+x_2}+\left(S_{in}-x_2\right)u\\ y&=x_1,\end{aligned}\tag{4.37}$$

which can be rearranged as:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}&=\begin{bmatrix}0 & \dfrac{\mu^* x_1}{K_c x_1+x_2}\\ 0 & -\dfrac{k\mu^* x_1}{K_c x_1+x_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-x_1\\ S_{in}-x_2\end{bmatrix}u\\ y&=x_1,\end{aligned}\tag{4.38}$$

where x_1 is the biomass concentration, x_2 the substrate, μ^* and K_c the Contois law parameters, and k a yield coefficient. The bioprocess inputs are the dilution rate u and the substrate concentration S_{in}.

Considering the state dependent terms in the matrices, i.e. $\zeta_1(x) = x_1$, $\zeta_2(x) = x_2$, and $\zeta_3(x) = \dfrac{x_1}{K_c x_1 + x_2}$, where x_2 is an unmeasured state, the following premise variables are defined according to representation (4.36): $z_\alpha = x_1$, $z_\beta^1 = x_2$, and $z_\beta^2 = \dfrac{x_1}{K_c x_1 + x_2}$. Thus, system (4.38) can be rewritten via the sector nonlinearity approach as a TS model (4.36) in the compact set $\mathcal{C}_x$ with the following matrices and MFs (matrices C, D, and G are constant):

$$A_{11} = A_{12} = A_{21} = A_{22} = \begin{bmatrix} 0 & \mu^* \underline{\zeta_3} \\ 0 & -k\mu^* \underline{\zeta_3} \end{bmatrix}, \qquad A_{13} = A_{23} = A_{14} = A_{24} = \begin{bmatrix} 0 & \mu^* \overline{\zeta_3} \\ 0 & -k\mu^* \overline{\zeta_3} \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$B_{11} = B_{13} = \begin{bmatrix} -\underline{\zeta_1} \\ S_{in} - \underline{\zeta_2} \end{bmatrix}, \quad B_{12} = B_{14} = \begin{bmatrix} -\underline{\zeta_1} \\ S_{in} - \overline{\zeta_2} \end{bmatrix}, \quad B_{21} = B_{23} = \begin{bmatrix} -\overline{\zeta_1} \\ S_{in} - \underline{\zeta_2} \end{bmatrix}, \quad B_{22} = B_{24} = \begin{bmatrix} -\overline{\zeta_1} \\ S_{in} - \overline{\zeta_2} \end{bmatrix},$$

$D = 0$, $G = 0$, $\omega_0^j(z(\cdot)) = \dfrac{\overline{\zeta_j} - \zeta_j}{\overline{\zeta_j} - \underline{\zeta_j}}$, $\omega_1^j(\cdot) = 1 - \omega_0^j(\cdot)$, $j \in \{1,2,3\}$, $\alpha_1 = w_0^1$, $\alpha_2 = w_1^1$, $\beta_1 = w_0^2 w_0^3$, $\beta_2 = w_1^2 w_0^3$, $\beta_3 = w_0^2 w_1^3$, $\beta_4 = w_1^2 w_1^3$, $h_1 = \alpha_1\beta_1$, $h_2 = \alpha_2\beta_1$, $h_3 = \alpha_1\beta_2$, $h_4 = \alpha_2\beta_2$, $h_5 = \alpha_1\beta_3$, $h_6 = \alpha_2\beta_3$, $h_7 = \alpha_1\beta_4$, $h_8 = \alpha_2\beta_4$.♦

Consider two vectors of errors: the usual state vector error $e = x - \hat{x}$, $e \in \mathbb{R}^{n_x}$, and the premise vector error concerned with the non-measurable variables $e_z = z_\beta - \hat{z}_\beta$, $e_z \in \mathbb{R}^{p_\beta}$. Also, a linear mapping between the non-measurable premise and the state vectors is considered hereafter, i.e., $z_\beta = Tx$, $T \in \mathbb{R}^{p_\beta \times n_x}$ (thus, $e_z = Te$). Extending the results to a class C^1 nonlinear mapping such as: $z_\beta : \mathbb{R}^{n_x} \to \mathbb{R}^{p_\beta}$ is straightforward; it will be discussed afterwards in section 4.3.3.

4.3.2. Observer design with unmeasured variables

The following observer structure is proposed for the TS model in (4.1):

$$\begin{aligned} \dot{\hat{x}} &= \sum_{i=1}^{r} h_i(\hat{z})\left(A_i\hat{x} + B_i u + P^{-1}K_i(y - \hat{y})\right) = A_{\hat{h}}\hat{x} + B_{\hat{h}}u + P^{-1}K_{\hat{h}}(y - \hat{y}) \\ \hat{y} &= \sum_{i=1}^{r} h_i(\hat{z}) C_i \hat{x} = C_{\hat{h}}\hat{x}, \end{aligned} \tag{4.39}$$

where $\hat{h}_i = h_i(\hat{z})$. Using the definitions presented after (4.36), the polytopic (TS) observer (4.39) can be written in a way that measured (z_α)/non-measured ($\hat{z}_\beta$) premise variables are separated:

$$\begin{aligned}\dot{\hat{x}} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\beta_j(\hat{z}_\beta)\left(A_{ij}\hat{x}+B_{ij}u+P^{-1}K_{ij}(y-\hat{y})\right)\\ &= A_{\alpha\hat{\beta}}\hat{x}+B_{\alpha\hat{\beta}}u+P^{-1}K_{\alpha\hat{\beta}}(y-\hat{y})\\ \hat{y} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\beta_j(\hat{z}_\beta)C_{ij}\hat{x} = C_{\alpha\hat{\beta}}\hat{x},\end{aligned} \tag{4.40}$$

with $\hat{x}\in\mathbb{R}^{n_x}$ as the observer state and $K_{\hat{h}} = K_{\alpha\hat{\beta}} \in \mathbb{R}^{n_x\times n_y}$ being matrix functions to be designed in the sequel. For the correspondence between both notations, it should be understood that $h=\alpha\beta$ and $\hat{h}=\alpha\hat{\beta}$, where $\hat{\beta}_j = \beta_j(\hat{z})$.

The following results are obtained using a quadratic Lyapunov function candidate of the form:

$$V(e)=e^T Pe,\ P=P^T>0. \tag{4.41}$$

Recalling that $e = x-\hat{x}$ is the estimation error and using (4.1) with (4.39), its dynamics is:

$$\dot{e} = A_h x - A_{\hat{h}}\hat{x} + B_h u - B_{\hat{h}}u + D_h w - P^{-1}K_{\hat{h}}\left(C_h x + G_h w - C_{\hat{h}}\hat{x}\right).$$

Provided $\hat{x} = x-e$, the previous expression is reorganized as:

$$\begin{aligned}\dot{e} = &\left(A_{\hat{h}} - P^{-1}K_{\hat{h}}C_{\hat{h}}\right)e + \left(A_h - A_{\hat{h}}\right)x + \left(B_h - B_{\hat{h}}\right)u\\ &- P^{-1}K_{\hat{h}}\left(C_h - C_{\hat{h}}\right)x + \left(D_h - P^{-1}K_{\hat{h}}G_h\right)w.\end{aligned} \tag{4.42}$$

To expand the grouped matrix differences above, the second notation is used:

$$\begin{aligned}A_{\alpha\beta} - A_{\alpha\hat{\beta}} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\left(\beta_j(z_\beta)-\beta_j(\hat{z}_\beta)\right)A_{ij},\\ B_{\alpha\beta} - B_{\alpha\hat{\beta}} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\left(\beta_j(z_\beta)-\beta_j(\hat{z}_\beta)\right)B_{ij},\\ C_{\alpha\beta} - C_{\alpha\hat{\beta}} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta}\alpha_i(z_\alpha)\left(\beta_j(z_\beta)-\beta_j(\hat{z}_\beta)\right)C_{ij}.\end{aligned} \tag{4.43}$$

Considering the differential mean value theorem in lemma B.2, and the premise error $e_z = z_\beta - \hat{z}_\beta$, (4.43) can be rewritten, i.e., it exists $c \in \left]z_\beta, \hat{z}_\beta\right[$ such that:

$$
\begin{aligned}
A_{\alpha\beta} - A_{\alpha\hat{\beta}} &= \sum_{i=1}^{r_\alpha}\sum_{j=1}^{r_\beta} \alpha_i(z_\alpha)\nabla\beta_j(c)e_z A_{ij} = \sum_{j=1}^{r_\beta}\nabla\beta_j(c)e_z A_{\alpha j}, \\
B_{\alpha\beta} - B_{\alpha\hat{\beta}} &= \sum_{j=1}^{r_\beta}\nabla\beta_j(c)e_z B_{\alpha j}, \quad C_{\alpha\beta} - C_{\alpha\hat{\beta}} = \sum_{j=1}^{r_\beta}\nabla\beta_j(c)e_z C_{\alpha j},
\end{aligned} \tag{4.44}
$$

with $\nabla\beta_j(c) = \dfrac{\partial\beta_j(c)}{\partial z_\beta}$.

Since $\sum_{j=1}^{r_\beta}\left(\beta_j(z_\beta) - \beta_j(\hat{z}_\beta)\right) = 0$, additional slack matrices can be introduced via algebraic equations. To do so, consider $Y_{\hat{h}\hat{h}} \in \mathbb{R}^{n_x \times n_x}$, $Z_{\hat{h}\hat{h}} \in \mathbb{R}^{n_x \times n_u}$, $Y_{\hat{h}\hat{h}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\hat{z})h_j(\hat{z})Y_{ij}$, and $Z_{\hat{h}\hat{h}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\hat{z})h_j(\hat{z})Z_{ij}$, so that:

$$
0 = \left(\sum_{j=1}^{r_\beta}\left(\beta_j(z_\beta) - \beta_j(\hat{z}_\beta)\right)\right)P^{-1}\left(Y_{\hat{h}\hat{h}}x + Z_{\hat{h}\hat{h}}u\right) = \left(\sum_{j=1}^{r_\beta}\nabla\beta_j(c)e_z\right)P^{-1}\left(Y_{\hat{h}\hat{h}}x + Z_{\hat{h}\hat{h}}u\right). \tag{4.45}
$$

Then, considering (4.44) and (4.45), the estimated error (4.42) writes:

$$
\begin{aligned}
\dot{e} &= \left(A_{\hat{h}} - P^{-1}K_{\hat{h}}C_{\hat{h}}\right)e + \left(D_h - P^{-1}K_{\hat{h}}G_h\right)w \\
&\quad + \sum_{j=1}^{r_\beta}\nabla\beta_j(c)e_z\left[\left(A_{\alpha j} - P^{-1}K_{\hat{h}}C_{\alpha j} + P^{-1}Y_{\hat{h}\hat{h}}\right)x + \left(B_{\alpha j} + P^{-1}Z_{\hat{h}\hat{h}}\right)u\right].
\end{aligned} \tag{4.46}
$$

With the assumption of a linear mapping $e_z = Te$ and considering the notation: $\overline{A}_{\hat{h}\hat{h}} = A_{\hat{h}} - P^{-1}K_{\hat{h}}C_{\hat{h}}$, $A_{\hat{h}\hat{h}j} = A_{\alpha j} - P^{-1}K_{\hat{h}}C_{\alpha j} + P^{-1}Y_{\hat{h}\hat{h}}$, $B_{\hat{h}\hat{h}j} = B_{\alpha j} + P^{-1}Z_{\hat{h}\hat{h}}$, $\overline{D}_{h\hat{h}} = D_h - P^{-1}K_{\hat{h}}G_h$, (4.46) yields:

$$
\dot{e} = \overline{A}_{\hat{h}\hat{h}}e + \overline{D}_{h\hat{h}}w + \sum_{j=1}^{r_\beta}\left[A_{\hat{h}\hat{h}j}x\nabla\beta_j(c) + B_{\hat{h}\hat{h}j}u\nabla\beta_j(c)\right]Te. \tag{4.47}
$$

Now considering that

$$
\sum_{j=1}^{r_\beta} A_{\hat{h}\hat{h}j}x\nabla\beta_j(c) = \begin{bmatrix} A_{\hat{h}\hat{h}1} & A_{\hat{h}\hat{h}2} & \cdots & A_{\hat{h}\hat{h}r_\beta}\end{bmatrix}\left(I_{r_\beta}\otimes x\right)\begin{bmatrix}\nabla\beta_1(c) \\ \nabla\beta_2(c) \\ \vdots \\ \nabla\beta_{r_\beta}(c)\end{bmatrix},
$$

with $\otimes$ standing for the classical Kronecker product, and similarly,

$$\sum_{j=1}^{r_\beta} B_{\hat{h}\hat{h}j} u \nabla \beta_j(c) = \begin{bmatrix} B_{\hat{h}\hat{h}1} & B_{\hat{h}\hat{h}2} & \cdots & B_{\hat{h}\hat{h}r_\beta} \end{bmatrix} \left(I_{r_\beta} \otimes u \right) \begin{bmatrix} \nabla \beta_1(c) \\ \nabla \beta_2(c) \\ \vdots \\ \nabla \beta_{r_\beta}(c) \end{bmatrix},$$

(4.47) is equivalent to:

$$\dot{e} = \bar{A}_{\hat{h}\hat{h}} e + \bar{D}_{h\hat{h}} w + N^a_{\hat{h}\hat{h}} \Delta_a Te + N^b_{\hat{h}\hat{h}} \Delta_b Te = \left(\bar{A}_{\hat{h}\hat{h}} + \bar{N}_{\hat{h}\hat{h}} \bar{\Delta} T \right) e + \bar{D}_{h\hat{h}} w, \tag{4.48}$$

with $\bar{N}_{\hat{h}\hat{h}} = \begin{bmatrix} N^a_{\hat{h}\hat{h}} & N^b_{\hat{h}\hat{h}} \end{bmatrix}$, $N^a_{\hat{h}\hat{h}} = \begin{bmatrix} A_{\hat{h}\hat{h}1} & A_{\hat{h}\hat{h}2} & \cdots & A_{\hat{h}\hat{h}r_\beta} \end{bmatrix}$, $N^b_{\hat{h}\hat{h}} = \begin{bmatrix} B_{\hat{h}\hat{h}1} & B_{\hat{h}\hat{h}2} & \cdots & B_{\hat{h}\hat{h}r_\beta} \end{bmatrix}$,

$\nabla \beta(c) = \begin{bmatrix} \nabla \beta_1(c) \\ \nabla \beta_2(c) \\ \vdots \\ \nabla \beta_{r_\beta}(c) \end{bmatrix}$, $\quad \bar{\Delta} = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix} \in \mathbb{R}^{((n_x+n_u)\cdot r)\times n_z}$, $\quad \Delta_a = (I_r \otimes x) \nabla \beta(c) = \begin{bmatrix} x\nabla \beta_1(c) \\ x\nabla \beta_2(c) \\ \vdots \\ x\nabla \beta_{r_\beta}(c) \end{bmatrix}$,

$\Delta_b = (I_r \otimes u) \nabla \beta(c) = \begin{bmatrix} u\nabla \beta_1(c) \\ u\nabla \beta_2(c) \\ \vdots \\ u\nabla \beta_{r_\beta}(c) \end{bmatrix}$.

Consider first the estimation without perturbation, $w = 0$:

$$\dot{e} = \left(\bar{A}_{\hat{h}\hat{h}} + \bar{N}_{\hat{h}\hat{h}} \bar{\Delta} T \right) e. \tag{4.49}$$

Let us recall the initial bounding assumptions: $\left\| \frac{\partial \beta_j}{\partial z_\beta} \right\| \le \sigma_j$ and $\|x\| \le \lambda_x$ and $\|u\| \le \lambda_u$. From $\left\| \frac{\partial \beta_j}{\partial z_\beta} \right\| \le \sigma_j$, it follows that $\nabla \beta(c) \nabla \beta^T(c) = \sum_{j=1}^{r_\beta} \nabla \beta_j(c) \nabla \beta_j^T(c) \le \sum_{j=1}^{r_\beta} \sigma_j^2$, and multiplying it by $(\lambda_x^2 + \lambda_u^2)$ gives $\left(\lambda_x^2 + \lambda_u^2\right) \nabla \beta(c) \nabla \beta^T(c) \le \eta^2$ with

$$\eta \ge \sqrt{\left(\lambda_x^2 + \lambda_u^2\right) \sum_{j=1}^{r_\beta} \sigma_j^2}. \tag{4.50}$$

The property B.4 states that $\left(\lambda_x^2 + \lambda_u^2\right) \nabla \beta^T(c) \nabla \beta(c) \le \eta^2 I_{n_z}$. Using the assumptions $\|x\| \le \lambda_x$ and $\|u\| \le \lambda_u$, it writes:

$$\nabla \beta^T(c) \left(I_r \otimes \left(x^T x + u^T u \right) \right) \nabla \beta(c) = \left(x^T x + u^T u \right) \nabla \beta^T(c) \nabla \beta(c) \le \eta^2 I_{n_z}.$$

Because $\nabla\beta^T(c)\left(\left(I_r \otimes x^T\right)\left(I_r \otimes x\right)+\left(I_r \otimes u^T\right)\left(I_r \otimes u\right)\right)\nabla\beta(c)=\Delta_a^T\Delta_a+\Delta_b^T\Delta_b=\overline{\Delta}^T\overline{\Delta}$, it is possible to find a bound of the uncertainties terms such that: $\overline{\Delta}^T\overline{\Delta}\le\eta^2 I_{n_z}$.

Theorem 4.7: The origin of the estimation error model (4.48) with $w=0$ is asymptotically stable for given scalars λ_x, λ_u, and σ_j, if there exist matrices $P=P^T>0$, K_{kl}, Y_{ijkl}, Z_{ijkl} of proper dimensions and scalars μ_{ijkl}, $i,k\in\{1,2,\ldots,r_\alpha\}$, $j,l\in\{1,2,\ldots,r_\beta\}$ such that the following LMI conditions hold:

$$\begin{aligned}&\Upsilon_{ijij}<0,\\&\frac{2}{r_\alpha-1}\Upsilon_{ijij}+\Upsilon_{ijkl}+\Upsilon_{klij}<0,\ i\neq k,\ j\neq l,\end{aligned}\tag{4.51}$$

with

$$\Upsilon_{ijkl}=\left[\begin{array}{c|c}\left(PA_{ij}-K_{kl}C_{ij}\right)+\left(PA_{ij}-K_{kl}C_{ij}\right)^T+\mu_{ijkl}\eta^2T^TT & (*)\\ \hline \left(PA_{i1}-K_{kl}C_{i1}+Y_{ijkl}\right)^T & \\ \vdots & \\ \left(PA_{ir_\beta}-K_{kl}C_{ir_\beta}+Y_{ijkl}\right)^T & -\mu_{ijkl}I_{r_\beta(n+m)}\\ \left(PB_{i1}+Z_{ijkl}\right)^T & \\ \vdots & \\ \left(PB_{ir_\beta}+Z_{ijkl}\right)^T & \end{array}\right],\ \eta\ge\sqrt{\left(\lambda_x^2+\lambda_u^2\right)\sum_{j=1}^{r_\beta}\sigma_j^2}\,.$$

Proof: Taking into account the estimation error model (4.49), the time-derivative of (4.41) gives:

$$\dot{V}(e)=e^T\left(P\overline{A}_{\hat{h}\hat{h}}+\overline{A}_{\hat{h}\hat{h}}^TP+P\overline{N}_{\hat{h}\hat{h}}\overline{\Delta}T+T^T\overline{\Delta}^T\overline{N}_{\hat{h}\hat{h}}^TP\right)e<0\,.\tag{4.52}$$

Applying property B.5, considering $\overline{\Delta}^T\overline{\Delta}\le\eta^2I_{n_z}$ and introducing $\mu_{\hat{h}\hat{h}}>0$, (4.52) holds if:

$$P\overline{A}_{\hat{h}\hat{h}}+\overline{A}_{\hat{h}\hat{h}}^TP+\mu_{\hat{h}\hat{h}}\eta^2T^TT+\mu_{\hat{h}\hat{h}}^{-1}P\overline{N}_{\hat{h}\hat{h}}\overline{N}_{\hat{h}\hat{h}}^TP<0\,.\tag{4.53}$$

Applying Schur complement to (4.53) gives:

$$\begin{bmatrix}P\overline{A}_{\hat{h}\hat{h}}+\overline{A}_{\hat{h}\hat{h}}^TP+\mu_{\hat{h}\hat{h}}\eta^2T^TT & P\overline{N}_{\hat{h}\hat{h}}\\ \overline{N}_{\hat{h}\hat{h}}^TP & -\mu_{\hat{h}\hat{h}}I_{r_\beta(n+m)}\end{bmatrix}<0\,.\tag{4.54}$$

Applying lemma C.6 to (4.54) leads to conditions defined in (4.51) which altogether with (4.50) guarantee the inequality above, thus producing the desired result. ◈

Remark 4.6: Conditions in Theorem 4.7 can be reduced to simpler conditions when matrices $B_{\hat{h}}$ and $C_{\hat{h}}$ have the following particular structures: a) $B_{\hat{h}}$ and $C_{\hat{h}}$ being constant, i.e., $B_1 = B_2 = \cdots = B_r = B$ and $C_1 = C_2 = \cdots = C_r = C$; b) $B_{\hat{h}}$ and $C_{\hat{h}}$ depending only on measurable premise variables, i.e., $B_{\hat{h}} = B_h$ and $C_{\hat{h}} = C_h$. Table 4.5 shows how conditions in Theorem 4.7 are simplified for these cases. Let us rewrite for clarity:

$$\Upsilon_{ijkl} = \begin{bmatrix} \left(PA_{ij} - K_{kl}C_{ij}\right) + (*) + \mu_{ijkl}\eta^2 T^T T & \Phi_{ijkl} \\ (*) & -\mu_{ijkl} I_{r_\beta(n+m)} \end{bmatrix},$$

$$\Phi_{ijkl} = \begin{bmatrix} PA_{i1} - K_{kl}C_{i1} + Y_{ijkl} & \ldots & PA_{ir_\beta} - K_{kl}C_{ir_\beta} + Y_{ijkl} & PB_{i1} + Z_{ijkl} & \ldots & PB_{ir_\beta} + Z_{ijkl} \end{bmatrix}.$$

Table 4.5. Particular cases

Case	Conditions with $i,k \in \{1,2,\ldots,r_\alpha\}$, $j,l \in \{1,2,\ldots,r_\beta\}$	Bound η	Eq.
$B_{\hat{h}} = B_h$ and $C_{\hat{h}} = C$	$\Upsilon_{ij} = \begin{bmatrix} \left(PA_{ij} - K_{ij}C\right) + (*) + \mu_{ij}\eta^2 T^T T & \Phi_{ij} \\ (*) & -\mu_{ij} I_{r_\beta n} \end{bmatrix}$ $\Phi_{ij} = \begin{bmatrix} PA_{i1} + Y_{ij} & \cdots & PA_{ir_\beta} + Y_{ij} \end{bmatrix}$	$\eta \ge \lambda_x \sqrt{\sum_{j=1}^{r_\beta} \sigma_j^2}$	(4.55)
$B_{\hat{h}} = B_h$ and $C_{\hat{h}} = C_h$	$\Upsilon_{ijkl} = \begin{bmatrix} \left(PA_{ij} - K_{ij}C_{kl}\right) + (*) + \mu_{ijkl}\eta^2 T^T T & \Phi_{ijkl} \\ (*) & -\mu_{ijkl} I_{r_\beta n} \end{bmatrix}$ $\Phi_{ijkl} = \begin{bmatrix} PA_{i1} + Y_{ijkl} & \cdots & PA_{ir_\beta} + Y_{ijkl} \end{bmatrix}$	$\eta \ge \lambda_x \sqrt{\sum_{j=1}^{r_\beta} \sigma_j^2}$	(4.56)
$C_{\hat{h}} = C$	$\Upsilon_{ij} = \begin{bmatrix} \left(PA_{ij} - K_{ij}C\right) + (*) + \mu_{ij}\eta^2 T^T T & \Phi_{ij} \\ (*) & -\mu_{ij} I_{r_\beta(n+m)} \end{bmatrix}$ $\Phi_{ij} = \begin{bmatrix} PA_{i1} + Y_{ij} & \cdots & PA_{ir_\beta} + Y_{ij} & PB_{i1} + Z_{ij} & \cdots & PB_{ir_\beta} + Z_{ij} \end{bmatrix}$	$\eta \ge \sqrt{\lambda_t \sum_{j=1}^{r_\beta} \sigma_j^2}$ $\lambda_t = \lambda_x^2 + \lambda_u^2$	(4.57)
$C_{\hat{h}} = C_h$	$\Upsilon_{ijkl} = \begin{bmatrix} \left(PA_{ij} - K_{ij}C_{kl}\right) + (*) + \mu_{ijkl}\eta^2 T^T T & \Phi_{ijkl} \\ (*) & -\mu_{ijkl} I_{r_\beta(n+m)} \end{bmatrix}$ $\Phi_{ijkl} = \begin{bmatrix} PA_{i1} + Y_{ijkl} & \cdots & PA_{ir_\beta} + Y_{ijkl} & PB_{i1} + Z_{ijkl} & \cdots & PB_{ir_\beta} + Z_{ijkl} \end{bmatrix}$	$\eta \ge \sqrt{\lambda_t \sum_{j=1}^{r_\beta} \sigma_j^2}$ $\lambda_t = \lambda_x^2 + \lambda_u^2$	(4.58)
$B_{\hat{h}} = B_h$	$\Upsilon_{ijkl} = \begin{bmatrix} \left(PA_{ij} - K_{kl}C_{ij}\right) + (*) + \mu_{ijkl}\eta^2 T^T T & \Phi_{ijkl} \\ (*) & -\mu_{ijkl} I_{r_\beta n} \end{bmatrix}$ $\Phi_{ijkl} = \begin{bmatrix} PA_{i1} - K_{kl}C_{i1} + Y_{ijkl} & \ldots & PA_{ir_\beta} - K_{kl}C_{ir_\beta} + Y_{ijkl} \end{bmatrix}$	$\eta \ge \lambda_x \sqrt{\sum_{j=1}^{r_\beta} \sigma_j^2}$	(4.59)

4.3.3. $H\infty$ disturbance rejection

The following theorem gives conditions to guarantee $H\infty$ attenuation criterion for the estimation error dynamics given in (4.27).

Theorem 4.8: The estimation error model (4.48) holds the $H\infty$ attenuation criterion (4.27) for given λ_x, λ_u, and σ_j, if there exist scalars $\mu_{ijkl}^m > 0$, $\gamma > 0$, and matrices $P = P^T > 0$, K_{kl}, Y_{ijkl}^m, Z_{ijkl}^m, $i,k \in \{1,2,\ldots,r_\alpha\}$, $j,l,m \in \{1,2,\ldots,r_\beta\}$ of proper dimensions such that the following LMI conditions hold

$$\begin{aligned} &\Upsilon_{ijij}^m < 0, \\ &\frac{2}{r-1}\Upsilon_{ijij}^m + \Upsilon_{ijkl}^m + \Upsilon_{klij}^m < 0, \quad i \neq k,\, j \neq l, \end{aligned} \tag{4.60}$$

with $\Upsilon_{ijkl}^m = \begin{bmatrix} (PA_{ij} - K_{kl}C_{ij}) + (*) + \mu_{ijkl}^m \eta^2 T^T T + I & \Phi_{ijkl}^m & PD_{im} - K_{kl}G_{im} \\ (*) & -\mu_{ijkl}^m I_{r_\beta(n_x+n_y)} & 0 \\ (*) & 0 & -\bar{\gamma} I_{n_w} \end{bmatrix}$, $\gamma = \sqrt{\bar{\gamma}}$,

$\Phi_{ijkl}^m = \begin{bmatrix} PA_{i1} - K_{kl}C_{i1} + Y_{ijkl}^m & \ldots & PA_{ir_\beta} - K_{kl}C_{ir_\beta} + Y_{ijkl}^m & PB_{i1} + Z_{ijkl}^m & \ldots & PB_{ir_\beta} + Z_{ijkl}^m \end{bmatrix}$, and $\eta \geq \sqrt{\left(\lambda_x^2 + \lambda_u^2\right)\sum_{j=1}^{r_\beta}\sigma_j^2}$.

Proof: It follows straightforwardly considering that (4.28) is:

$$\begin{bmatrix} e \\ w \end{bmatrix}^T \begin{bmatrix} P\bar{A}_{\hat{h}\hat{h}} + \bar{A}_{\hat{h}\hat{h}}^T P + P\bar{N}_{\hat{h}\hat{h}}\bar{\Delta}T + T^T\bar{\Delta}^T\bar{N}_{\hat{h}\hat{h}}^T P + I & P\bar{D}_{h\hat{h}} \\ \bar{D}_{h\hat{h}}^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} < 0;$$

then following the same path as in Theorem 4.7 to obtain:

$$\begin{bmatrix} P\bar{A}_{\hat{h}\hat{h}} + \bar{A}_{\hat{h}\hat{h}}^T P + \mu_{\hat{h}\hat{h}}\eta^2 T^T T + \mu_{\hat{h}\hat{h}}^{-1} P\bar{N}_{\hat{h}\hat{h}}\bar{N}_{\hat{h}\hat{h}}^T P + I & P\bar{D}_{h\hat{h}} \\ \bar{D}_{h\hat{h}}^T P & -\gamma^2 I \end{bmatrix} < 0. \tag{4.61}$$

Applying Schur complement and lemma C.6 with $\gamma = \sqrt{\bar{\gamma}}$ and $\eta \geq \sqrt{\left(\lambda_x^2 + \lambda_u^2\right)\sum_{j=1}^{r_\beta}\sigma_j^2}$, the proof is complete. ◈

Remark 4.7: In order to compare with results in (Ichalal et al., 2011), the following conditions are obtained for the particular case $B_{\hat{h}} = B_h$ and $C_{\hat{h}} = C$:

$$\Upsilon_{ii}^{jl} < 0,\ \forall i \in \{1,2,\ldots,r_\alpha\},\ \ \forall j,l \in \{1,2,\ldots,r_\beta\},$$
$$\frac{2}{r_\alpha - 1}\Upsilon_{ii}^{jl} + \Upsilon_{ik}^{jl} + \Upsilon_{ki}^{jl} < 0,\ \ \forall i,k \in \{1,2,\ldots,r_\alpha\},\ \ \forall j,l \in \{1,2,\ldots,r_\beta\}, \tag{4.62}$$

with $$\Upsilon_{ik}^{jl} = \begin{bmatrix} \left(PA_{ij} - K_{ij}C\right) + (*) + \mu_{ijkl}\eta^2 T^T T + I_{n_x} & \Phi_{ijkl} & PD_{kl} - K_{ij}G_{kl} \\ (*) & -\mu_{ijkl} I_{r_\beta(n_x+n_u)} & 0 \\ (*) & 0 & -\bar{\gamma} I_{n_w} \end{bmatrix}, \qquad \gamma = \sqrt{\bar{\gamma}},$$

$\Phi_{ijkl} = \begin{bmatrix} PA_{i1} + Y_{ijkl} & \ldots & PA_{ir_\beta} + Y_{ijkl} \end{bmatrix}$, and $\eta \geq \lambda_x \sqrt{\sum_{j=1}^{r_\beta} \sigma_j^2}$.

Remark 4.8 (generalization): When a nonlinear mapping $z_\beta : \mathbb{R}^{n_x} \to \mathbb{R}^{p_\beta}$ of class C^1 is used, the mean value theorem can still be used in the following way: it exists $c = \rho x + (1-\rho)\hat{x}$, $\rho \in \left]0,1\right[$ such that:

$$\beta_j \circ z_\beta(x) - \beta_j \circ z_\beta(\hat{x}) = \left.\frac{\partial \beta_j \circ z_\beta(x)}{\partial x}\right|_{x=c} (x - \hat{x}) = \nabla\left(\beta_j \circ z_\beta(c)\right) e, \tag{4.63}$$

from which the bounds in $\mathcal{C}_x$: $\left\| \frac{\partial\left(\beta_j \circ z_\beta(x)\right)}{\partial x} \right\| \leq \sigma_j$ can be defined. Therefore, the results are the same considering the new bounds and $T = \frac{\partial z_\beta}{\partial x}$, $\frac{\partial\left(\beta_j \circ z_\beta(x)\right)}{\partial x} = \frac{\partial\left(\beta_j(z_\beta)\right)}{\partial z_\beta} \times \frac{\partial z_\beta}{\partial x}$. For instance, for $\beta_1 \circ z_\beta(x) = \sin(x_1 x_2)$ we have $\frac{\partial\left(\beta_1(z_\beta)\right)}{\partial z_\beta} = \cos(z_\beta)$; altogether with 3 states it follows that $T = \frac{\partial z_\beta}{\partial x} = \begin{bmatrix} x_2 & x_1 & 0 \end{bmatrix}$.

Example 4.2 (continued): An exact polytopic (TS) model as in (4.36) for the compact set $\mathcal{C}_x = \left\{x : x_1 \in [0.01, 0.2], x_2 \in [0, 0.1]\right\}$ can be obtained for $\mu^* = 1$, $k = 1$, $K_c = 1000$, $S_{in} = 0.1$:

$A_{11} = A_{12} = A_{21} = A_{22} = \begin{bmatrix} 0 & 0.00099 \\ 0 & -0.00099 \end{bmatrix}$, $A_{13} = A_{14} = A_{23} = A_{24} = \begin{bmatrix} 0 & 0.001 \\ 0 & -0.001 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

$B_{11} = B_{13} = \begin{bmatrix} -0.01 \\ 0.1 \end{bmatrix}$, $B_{12} = B_{14} = \begin{bmatrix} -0.01 \\ 0 \end{bmatrix}$, $B_{21} = B_{23} = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}$, $B_{22} = B_{24} = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}$, $D = 0$, $G = 0$.

Since the MFs depend on the unmeasured state x_2, it is not possible to construct an "exact" observer (in the sense of the sector nonlinearity approach) with the classical measurable-premise approach for TS models.

In this example, $u = 0.01\left(1+\sin\left(0.2t\right)\right)$, $\lambda_x^2 = 0.0501$, $\lambda_u^2 = \left(0.02\right)^2$ ($x^T x \le \lambda_x^2$ and $u^T u \le \lambda_u^2$), $\sigma_1^2 = 726.98$, $\sigma_2^2 = 10199$, $\sigma_3^2 = 1029$, $\sigma_4^2 = 9903$ (so $\nabla\beta_i \nabla\beta_i^T \le \sigma_i^2$), which gives $\eta = 33.22$.

A feasible solution is obtained with the new conditions (4.57). The Lyapunov matrix and observer gains are $P = \begin{bmatrix} 6064613 & -957.28 \\ -957.28 & 0.1550 \end{bmatrix}$, $K_{11} = K_{12} = K_{21} = K_{22} = \begin{bmatrix} 4675985 \\ 4599 \end{bmatrix}$, and $K_{13} = K_{14} = K_{23} = K_{24} = \begin{bmatrix} 4619681 \\ 4677 \end{bmatrix}$.

The estimated state $\hat{x}_2$ is compared with the real state x_2 in Figure 4.6; also, the estimation error e_2 is presented in Figure 4.7: both figures confirm the effectiveness of the theoretical results. The initial conditions used in this example are $x^T\left(0\right) = \left[0.2 \quad 0.05\right]^T$ and $\hat{x}^T\left(0\right) = \left[0.2 \quad 0\right]^T$.♦

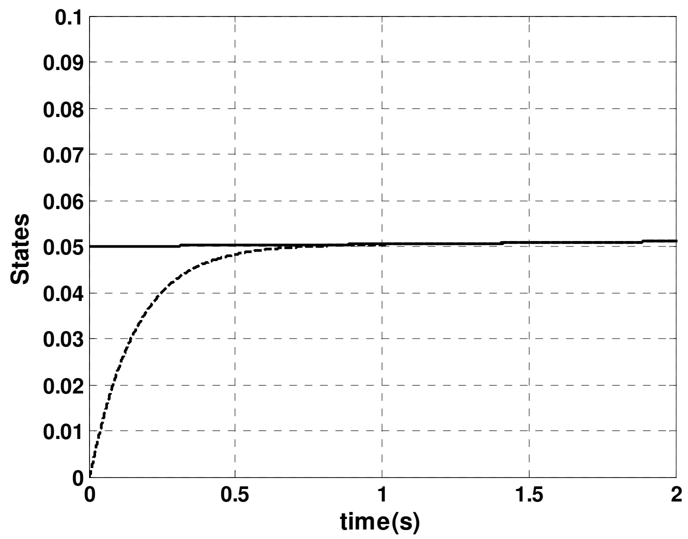


Figure 4.6. Time evolution of the states: x_2 (solid line) and $\hat{x}_2$ (dashed line).

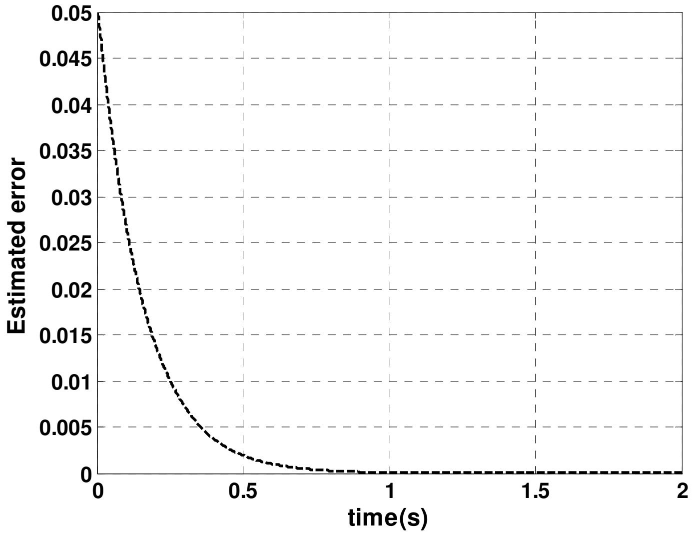


Figure 4.7. Estimation error e_2.

Example 4.3: Consider the following nonlinear model:

$$\dot{x} = \begin{bmatrix} \frac{a}{4}(x_1 + x_2) & b-3 \\ \frac{3}{4}(x_1 + x_2) & x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (4.64)$$
$$y = x_1.$$

Assuming that nonlinearities are not measurable, this model can be turned into the TS form (4.36) for $\mathcal{C}_x = \{x : |x_1 + x_2| \le 4\}$, with $A_{11} = \begin{bmatrix} -a & b-3 \\ 3 & -4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} a & b-3 \\ -3 & 4 \end{bmatrix}$, $B_{11} = B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0]$, $\beta_1 = \frac{4 - z_\beta}{8}$, $\beta_2 = 1 - \beta_1$, $z_\beta = x_1 + x_2$, $e_z = z_\beta - \hat{z}_\beta$, and $z_\beta = [1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx$, from which the premise error is $e_z = (x_1 + x_2) - (\hat{x}_1 + \hat{x}_2) = [1 \quad 1]\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = Te$.

In this example, $\nabla\beta_1(c) = \left.\frac{\partial \beta_1(z_\beta)}{\partial z_\beta}\right|_{z_\beta = c} = \frac{\partial}{\partial z_\beta}\left(\frac{4 - z_\beta}{8}\right) = -\frac{1}{8}$, $\nabla\beta_2(c) = -\nabla\beta_1(c) = \frac{1}{8}$, with bounds $\lambda_x^2 = 8$ ($x^T x \le \lambda_x^2$) and $\sigma_1^2 = \sigma_2^2 = 0.0156$ ($\nabla\beta_i \nabla\beta_i^T \le \sigma_i^2$), which give $\eta = 0.5$.

Figure 4.8 shows the feasibility regions obtained by (a) varying parameters $a \in [-2, 2]$ and $b \in [-2, 3]$ under conditions (4.55) in Table 4.5, (b) using Theorem 3 in (Bergsten et al., 2001), and (c) using Theorem 1 in (Ichalal et al., 2007), (b) and (c) being Lipchitz-based conditions. Parameter μ for conditions in (Bergsten et al., 2001) is calculated in each iteration while for Theorem 1 in (Ichalal et al., 2007) parameters $m_1 = m_2 = 1.5$, $n_1 = n_2 = 0.125$, and $\beta_1 = 0$ are chosen. The solution set of the new approach includes those in former ones. ♦

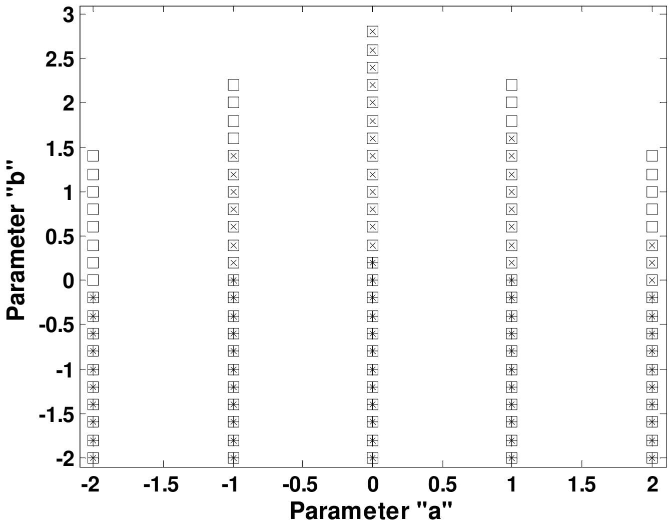


Figure 4.8. Feasibility region: "□" for conditions (4.55), "×" for conditions in (Ichalal et al., 2007), and "+" for conditions in (Bergsten et al., 2001).

Example 4.4: Consider the following nonlinear model:

$$\begin{aligned}
\dot{x}_1 &= -1.5x_1 - 0.5x_1^3 + 0.5ax_2 - 1.5x_2 \sin\left(x_1^2 + x_2\right) + \left(0.1 + 0.1\sin\left(x_1^2 + x_2\right)\right)w, \\
\dot{x}_2 &= 2x_1 + \left(b - 2\right)x_2 + x_2 \sin\left(x_1^2 + x_2\right) - \left(\frac{1}{1+10\phi} + 0.1\sin\left(x_1^2 + x_2\right)\right)w + u, \\
y &= x_1,
\end{aligned} \tag{4.65}$$

which can be rewritten as:

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -1.5 - 0.5\xi_1(x) & 0.5a - 1.5\xi_2(x) \\ 2 & b - 2 + \xi_2(x) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.1 + 0.1\xi_2(x) \\ -\dfrac{1}{1+10\phi} - 0.1\xi_2(x) \end{bmatrix} w, \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x,
\end{aligned} \tag{4.66}$$

with $\xi_1(x) = x_1^2$ and $\xi_2(x) = \sin\left(x_1^2 + x_2\right)$, where x_2 is an unmeasured state; the premise vector with measured (unmeasured) state variables has entries $z_\alpha = x_1^2$ ($z_\beta = x_1^2 + x_2$). Then, a TS model (4.36) in the compact set $\mathcal{C}_x = \left\{x \in \mathbb{R}^2 : |x_1| \le 1\right\}$ can be obtained where:

$A_{11} = \begin{bmatrix} -1.5 & 0.5a+1.5 \\ 2 & b-3 \end{bmatrix}$, $A_{21} = \begin{bmatrix} -2 & 0.5a+1.5 \\ 2 & b-3 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -1.5 & 0.5a-1.5 \\ 2 & b-1 \end{bmatrix}$,

$A_{22} = \begin{bmatrix} -2 & 0.5a-1.5 \\ 2 & b-1 \end{bmatrix}$, $B_{11} = B_{12} = B_{21} = B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D_{11} = D_{21} = \begin{bmatrix} 0 \\ 0.1 - \dfrac{1}{1+10\phi} \end{bmatrix}$,

$D_{12} = D_{22} = \begin{bmatrix} 0.2 \\ -0.1 - \dfrac{1}{1+10\phi} \end{bmatrix}$, $G = 0$, $\phi \in [0,1]$, $\beta_1(z_\beta) = \dfrac{1-\sin(z_\beta)}{2}$, $\beta_2(z_\beta) = \dfrac{1+\sin(z_\beta)}{2}$,

$\alpha_1(z_\alpha) = 1 - z_\alpha$, $\alpha_2(z_\alpha) = z_\alpha$, $h_1(z) = \alpha_1(z_\alpha)\beta_1(z_\beta)$, $h_2(z) = \alpha_2(z_\alpha)\beta_1(z_\beta)$, $h_3(z) = \alpha_1(z_\alpha)\beta_2(z_\beta)$, $h_4(z) = \alpha_2(z_\alpha)\beta_2(z_\beta)$.

No disturbances are considered for this example, i.e., $w = 0$. The following parameters are calculated:

$\nabla\beta_1(c) = \left.\dfrac{\partial\beta_1(z_\beta)}{\partial z_\beta}\right|_{z_\beta = c} = \dfrac{\partial}{\partial z_\beta}\left(\dfrac{1-\sin(z_\beta)}{2}\right) = \left| -\dfrac{\cos(c)}{2} \right| \leq \dfrac{1}{2}$, $\nabla\beta_2(c) = -\nabla\beta_1(c) \leq \dfrac{1}{2}$, with bounds $\lambda_x^2 = 1 + \pi^2$ ($x^T x \leq \lambda_x^2$) and $\sigma_1^2 = \sigma_2^2 = 0.25$ (so $\nabla\beta_i \nabla\beta_i^T \leq \sigma_i^2$), which gives $\eta = \sqrt{\dfrac{1+\pi^2}{2}}$. Also, $e_z = z_\beta - \hat{z}_\beta = x_1^2 + x_2 - \hat{x}_1^2 - \hat{x}_2$ and because x_1 is a measured variable in this example, then $e_z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = Te$.

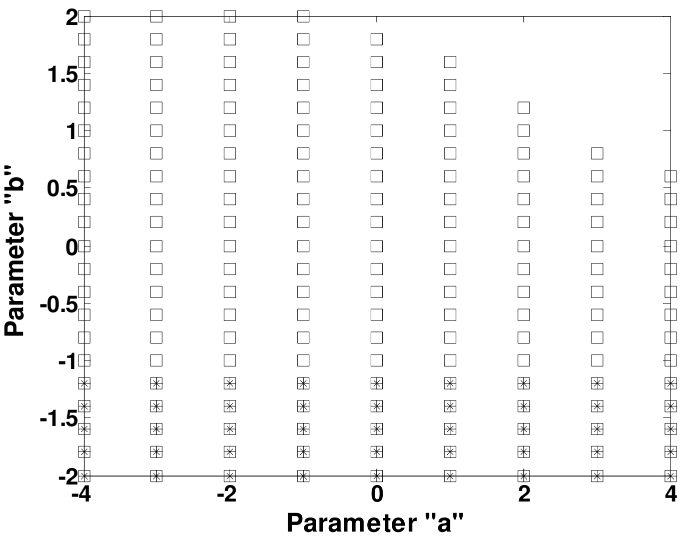


Figure 4.9. Feasibility region: "□" for conditions (4.55), "∗" for conditions in (Ichalal et al., 2011).

Figure 4.9 shows the feasibility sets for parameters $a \in [-4,4]$ and $b \in [-2,2]$ when conditions (4.55) are compared with those in Theorem 2 of (Ichalal et al., 2011), which also uses the mean value theorem. Once again, the new conditions provide significantly larger feasibility sets.

Now, consider the particular case of $a = 1$ and $b = -1$. Using conditions (4.55) for state estimation with unmeasured variables, a feasible solution is obtained with $\mu_{11} = \mu_{21} = 1.0236$ and $\mu_{12} = \mu_{22} = 0.9616$; on the other hand, conditions in Theorem 2 of (Ichalal et al., 2011) are rendered unfeasible. The corresponding Lyapunov matrix and observer gains are given by:

$$P = \begin{bmatrix} 0.5703 & 0.7258 \\ 0.7258 & 1.1844 \end{bmatrix}, \quad K_{11} = \begin{bmatrix} 1.1491 \\ -0.5241 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} 1.1558 \\ -0.7917 \end{bmatrix}, \quad K_{21} = \begin{bmatrix} 0.8640 \\ -0.8870 \end{bmatrix},$$

$$K_{22} = \begin{bmatrix} 0.8707 \\ -1.1545 \end{bmatrix}.$$

In order to simulate this example, a PDC control law $u(t) = F_{\alpha\beta}x(t) + 0.4\sin(10t)$ will be applied to stabilize the nonlinear system; the following controller gains are used: $F_{11} = [-16.8059 \quad 3.8224]$, $F_{12} = [5.4607 \quad 1.7553]$, $F_{21} = [-16.7947 \quad 3.8223]$, $F_{22} = [5.4719 \quad 1.7552]$.

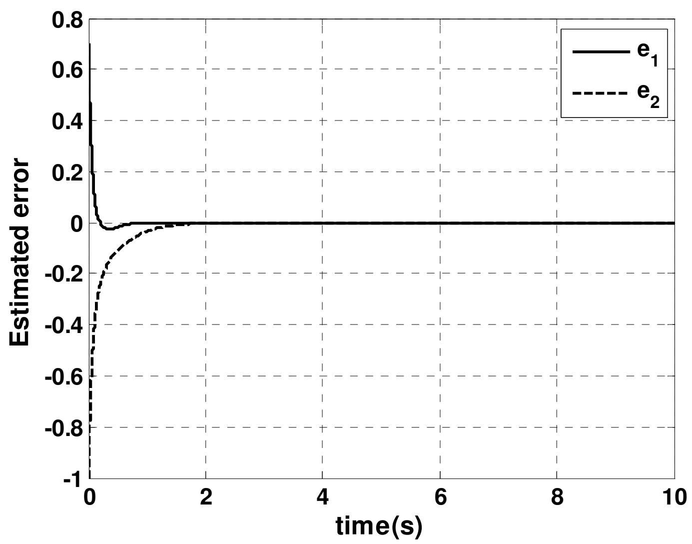


Figure 4.10. Time evolution of the estimation error: e_1 (solid line) and e_2 (dashed line).

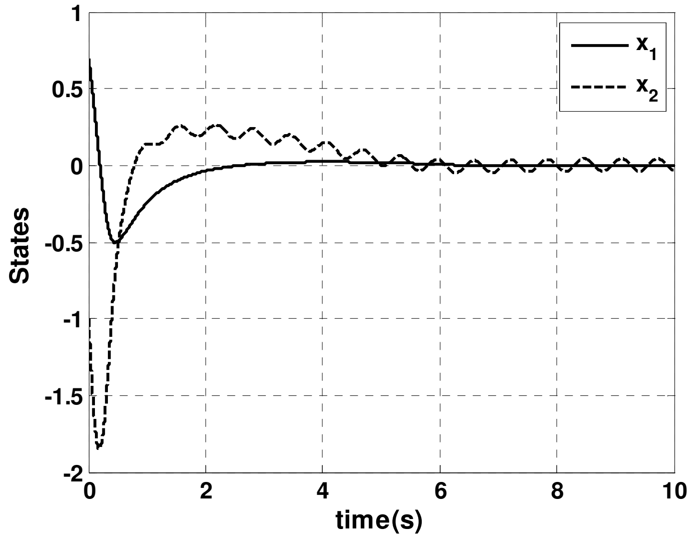


Figure 4.11. Time evolution of the states: x_1 (solid line) and x_2 (dashed line).

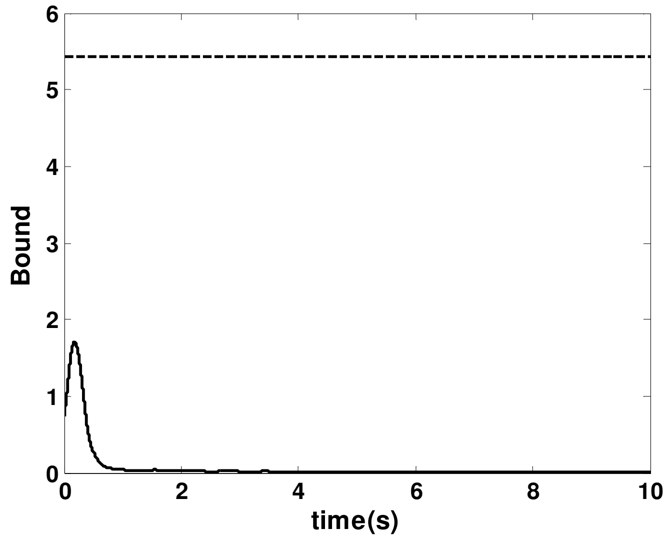


Figure 4.12. Bound: $\eta^2 I \geq \overline{\Delta}^T \overline{\Delta}$, ($\overline{\Delta}^T \overline{\Delta}$ solid line, η^2 dashed line).

Time evolution of the estimation errors under initial conditions $x(0) = \begin{bmatrix} 0.7 & -1 \end{bmatrix}^T$ and $\hat{x}^T(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is shown in Figure 4.10; the corresponding states are shown in Figure 4.11: as expected, despite the fact that they keep oscillating, the observer does not lose track of the observed system. In order to illustrate the fact that $\overline{\Delta}^T \overline{\Delta} \leq \eta^2 I$ is satisfied over time, Figure 4.12 is also included. ♦

Example 4.4 (continued): Now, consider the TS model with $a = 3$, $b = -2$, and $w(t) \neq 0$. The minimum performance bound γ obtained by Theorem 3 in (Ichalal et al., 2011) and by conditions (4.62) in Remark 4.7, for $\phi \in [0,1]$, is shown in Figure 4.13.

Conditions in (4.62) clearly outperform those in (Ichalal et al., 2011) by leading to remarkably lower attenuation minima.♦

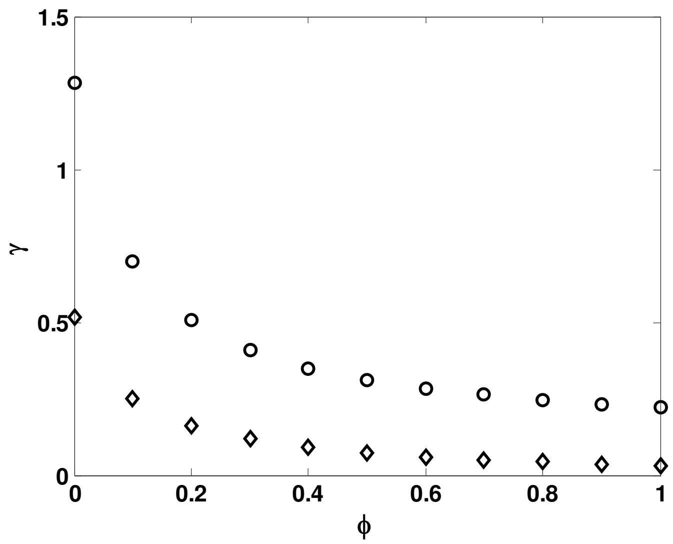


Figure 4.13. γ values: "o" for Th. 3 in (Ichalal et al., 2011) and "◊" for conditions (4.62).

4.4. Concluding Remarks.

Novel approaches for observer design of continuous-time nonlinear models that overcome up-to-date results in the state of the art, have been reported. A first set of results focuses on observer design based on measurable premises: taking advantage of a convex rewriting of the model (TS form) as well as from several matrix transformations such as the Finsler's Lemma, the observer is decoupled from their corresponding Lyapunov function; additionally, the proposed decoupling allows introducing progressively better results thanks to a nested convex structure. A second set of solutions in this chapter considers the unmeasured-premise case: the state estimation problem is expressed as a convex one using a quadratic Lyapunov function and the differential mean value theorem; the structure thus obtained allows introducing slack variables. Both the measured- and unmeasured-premise case include extensions on $H\infty$ disturbance rejection, where better results than those already available in the literature are reported.

Chapter 5. Conclusions and future work

This chapter gives some general concluding remarks on the results presented in this thesis as well as possible directions for future research.

5.1. General conclusions

This thesis has been directed on the analysis of continuous-time nonlinear systems via Takagi-Sugeno models. The following TS representations have been considered along this thesis: 1) standard TS model (5.1) and 2) descriptor TS model (5.2).

$$\text{Standard TS model: } \dot{x} = \sum_{i=1}^{r} h_i(z)(A_i x + B_i u), \; y = \sum_{i=1}^{r} h_i(z)(C_i x). \tag{5.1}$$

$$\text{Descriptor TS model: } \sum_{k=1}^{r} v_k(z) E_k \dot{x} = \sum_{i=1}^{r} h_i(z)(A_i x + B_i u), \; y = \sum_{i=1}^{r} h_i(z)(C_i x). \tag{5.2}$$

The following problems have been addressed:

- State feedback controller design.
- Observer design.

The goal of all new approaches has been focused in to reduce the conservativeness on the conditions of former results within TS-LMI framework.

New schemes for *state feedback controller design* have been developed for the TS representations (5.1) and (5.2). For the standard case, these schemes which are based on some matrix properties can be split in two parts: 1) via quadratic Lyapunov functions and 2) via non-quadratic Lyapunov functions. For the first part, the scheme proposed incorporates a relaxation based on multiple convex sums such that asymptotic features are kept. For the second part, schemes based on fuzzy (FLF) or line-integral (LILF) Lyapunov functions have been presented. The conditions thus achieved are *local* or *global* within a compact set of the state space for FLF and LILF, respectively. For stabilization problem based on a LILF, convex structure remains until second order systems. In addition, a new non-quadratic Lyapunov functional (NQLF) has been proposed which leads to less conservative conditions. For descriptor case, the schemes developed have been based on QLF as well as LILF. Also, the disturbance rejection problem has been addressed. All of these strategies give more relaxed conditions respect to previous works.

In the case of *observer design* only standard TS model (5.1) has been considered. Two directions have been discussed: 1) premise vectors are based on measured variables and 2) premise vectors are based on unmeasured variables or both. In the first direction progressively more relaxed conditions via a quadratic Lyapunov function and multiple nested sums have been obtained. Some alternatives through matrix transformations have been given. For the second direction, a novel scheme based on the differential mean value theorem has been addressed. The feasibility set of this approach overcomes the results of previous methods on this direction. The LMI conditions obtained assure asymptotical convergence to zero for the state estimation error. An extension of these results to $H\infty$ performance design has been presented. The contributions developed for observer/controller design are summarized in the following tables.

Table 5.1. Contributions on controller design

Controller design	**QLF**	**FLF**	**LILF**	**NQLF**
Standard TS model	Generalized approaches: - Technical lemmas - Multiple convex sums	Generalized approach	LMIs for second order systems	LMIs under a new Lyapunov Functional
Descriptor TS model	Parameter dependent LMIs: - Extended control law - Finsler lemma	-	LMIs for second order systems	-

Table 5.2. Contributions on observer design

Observer design	**Measured premise variables**	**Unmeasured premise variables**
QLF	Generalized approaches: - Technical lemmas - Multiple convex sums	LMIs based on mean value theorem

5.2. Future work

This section gives some directions for future research.

5.2.1. Stabilization with multiple nested convex sums

When the following control law is adopted:

$$u = F_{hh\cdots h} H_{hh\cdots h}^{-1} x, \tag{5.3}$$

where $F_{\underbrace{hh\cdots h}_{q}}$ and $H_{\underbrace{hh\cdots h}_{q}}$ are convex matrices and " q " is the number of convex sums.

Whatever is the approach used altogether with the complex control law (5.3), it can lead to problems for implementation purposes because several multiplications and matrix inversion have to be performed on real-time applications, especially in nonlinear TS representations with a high number of rules. However, some tracks can be explored. Once we get a stabilizing Lyapunov function, it is possible to: 1) apply regressions on the control law to get a simpler one (approximation); 2) build a numerical map (may use a lot of memory); 3) use the obtained Lyapunov function and search for a simpler controller by LMI (may lead to infeasibility if the problem is sensitive) or via optimal control (may lead to high gains).

5.2.2. Non-quadratic Lyapunov functions

An interesting result for stabilization which circumvent handling the time-derivatives of MFs, are based on a *line-integral Lyapunov function* which leads to global conditions within the compact set of the state space (Rhee and Won, 2006). This approach has been improved at least for second order systems leading to LMI conditions instead of BMI. Nevertheless, if higher order systems are considered, then the problem cannot be treated as a convex problem. In order to understand the difficulties that arise with the control problem associated with $V(x) = 2\int_{\Gamma(0,x)} f(\psi)\,d\psi$, $f(\psi) = P_h^{-1}x$, just consider the case of a 2-rules third order TS model. In this case, the Lyapunov function must verify the path-independent conditions (see chapter 3), which means that the matrix of the Lyapunov function has the following form q_{ij}, $j > i$ and d_i are the unknown variables:

$$P_h^{-1} = \begin{bmatrix} d_\omega & q_{12} & q_{13} \\ q_{12} & d_2 & q_{23} \\ q_{13} & q_{23} & d_3 \end{bmatrix}, \tag{5.4}$$

Of course, the problem comes from the fact that the LMI constraints are written according to P_h, i.e. with $P_h = \frac{1}{\left|P_h^{-1}\right|}\begin{bmatrix} d_2 d_3 - q_{23}^2 & q_{13}q_{23} - q_{12}d_3 & q_{12}q_{23} - q_{13}d_2 \\ q_{13}q_{23} - q_{12}d_3 & d_{\omega 1}d_3 - q_{13}^2 & q_{12}q_{13} - q_{23}d_{\omega 1} \\ q_{12}q_{23} - q_{13}d_2 & q_{12}q_{13} - q_{23}d_{\omega 1} & d_{\omega 1}d_2 - q_{12}^2 \end{bmatrix}$. Despite of the fact that $\left|P_h^{-1}\right|$ can be easily removed from the LMI constraints (the same way as for the 2nd order case), the adjoin matrix presents BMI conditions for the 3rd order and polynomial conditions for higher order and up-to-now no transformation is available to render the problem LMI.

Therefore, tracks that can be pursued need some extra matrix operations, such that Finsler's lemma or the use of descriptor redundancy. Another point of view could be to solve a sort of 2 path algorithm. The first one would consider the solution to the general problem $A_h P_h + B_h F_h + (*) < 0$ with P_h holding the path-independent structure. Note that as duality is not satisfied, this does not guarantee that the "real" problem involving path-independent structure of P_h^{-1} will be satisfied. This point is illustrated via counter-example in (Guelton et al., 2010). If a solution is found, then use the gains obtained F_h in a single stability problem of the closed-loop that is a LMI constraints problem such as in (Rhee and Won, 2006).

5.2.3. Non-quadratic Lyapunov functional

A new approach via a *non-quadratic Lyapunov functional* (NQLF) have been proposed which gives also global conditions without imposing strong restrictions on the structure of MFs as path-independent line integrals do and without any limitation on the system order. The NQLF has the following structure:

$$V(x) = x^T P_s^{-1} x = x^T \left(\sum_{i=1}^{r} s_i\left(z(t)\right) P_i \right)^{-1} x, \tag{5.5}$$

with $P_i = P_i^T > 0$, and

$$s_i\left(z(t)\right) = \frac{1}{\alpha}\int_{t-\alpha}^{t} h_i\left(z(\tau)\right) d\tau \geq 0,\ \alpha > 0. \tag{5.6}$$

Following the same generalization that can be found in (Ding, 2010, Lee et al., 2011, Estrada-Manzo et al., 2015) with multiple sums Lyapunov functions, (5.5) could be easily generalized to functional such that:

$$V(x) = x^T P^{-1}_{\underbrace{s\ldots s}_{q}} x = x^T \left(\sum_{i_1=1}^{r} \cdots \sum_{i_q=1}^{r} s_{i_1}(z) \ldots s_{i_q}(z) P_{i_1 \ldots i_q} \right)^{-1} x , \tag{5.7}$$

where all the $s_{i_j}(z)$, $j \in \{1,2,\ldots,q\}$ corresponds to definition (5.6), or even defining different delays α_j, $j \in \{1,2,\ldots,q\}$:

$$s_{i_j}(z(t)) = \frac{1}{\alpha_j} \int_{t-\alpha_j}^{t} h_{i_j}(z(\tau)) d\tau \geq 0 , \ \alpha_j > 0 . \tag{5.8}$$

Therefore, these extensions also directly make sense for TS time-delay systems since the Lyapunov functional does already implicitly involve delays which could be easily to handle.

Consider the following classical Lyapunov-Krasovskii functional candidate for time-delay systems (Gu et al., 2003):

$$V = \underbrace{x^T P x}_{V_1} + \underbrace{\int_{t-\tau}^{t} x^T(s) S x(s) ds}_{V_2} + \underbrace{\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta}_{V_3} , \tag{5.9}$$

where τ is the time-delay. The idea is to substitute the term V_1 in (5.9) by the NQLF in (5.5) or its extension (5.7), and try to get LMI conditions improving former results.

5.2.4. Observer design

All approaches presented in this thesis for state estimation assume a quadratic Lyapunov function for both cases measured/unmeasured variables premise vector. Therefore, non-quadratic Lyapunov functions or functional can be applied in the analysis in order to reduce the degree of sufficiency of the LMI conditions. Nevertheless, this problem is far from being easy. In the "simplest" case the Lyapunov function writes:

$$V(e) = e^T P_h e = x^T \sum_{i=1}^{r} h_i(z(t)) P_i x , \tag{5.10}$$

with the state error dynamic $e = x - \hat{x}$ in the form:

$$\dot{e} = \left(A_h - P_h^{-1} K_h C_h \right) e . \tag{5.11}$$

Then naturally the derivative of the Lyapunov function will include $P_{\dot{h}}$ which writes (see chapter 2):

$$P_{\dot{h}} = \sum_{i=1}^{r} \sum_{k=1}^{p} h_i \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(i,k)} - P_{g_2(i,k)} \right) \dot{z}_k , \tag{5.12}$$

with $g_1(i,k) = \lfloor (i-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (i-1) \bmod 2^{p-k}$ and $g_2(i,k) = g_1(i,k) + 2^{p-k}$, $\lfloor \bullet \rfloor$ stands for the floor function. Therefore, the derivative of the Lyapunov function explicitly includes $\dot{z}_k(t)$, which means a part of $\dot{x}(t)$. Thus, it is directly impossible to cope with the observation of an unstable system, as to cope with (5.12), it will be required to have $\|\dot{z}_k(t)\|$ bounded. This point can be thought as a serious limitation.

On other hand, the results presented in this work about state estimation are only developed for standard TS model, thus, it would be interesting to extend them for descriptor TS representation.

5.2.5. Real-time applications

All results provided in this thesis are under a theoretical framework. Nevertheless, it is also important to test the different approaches provided in this work for different physical systems and compare with other strategies as heuristic fuzzy controller (model-free) or classical methodologies on nonlinear control. Some academic physical systems that are available at ITSON which could be used are: the inverted pendulum rail, the furuta pendulum, twin rotor MIMO, interconnected tanks, and 3D crane.

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APPENDIX A. Lyapunov Stability

A.1. Stability Criteria

The following definitions an theorems can be found in (Khalil, 2002). Consider the autonomous nonlinear system:

$$\dot{x} = f(x), \ x \in \mathcal{D}, \tag{A.1}$$

with an equilibrium point in $x = 0$ where $\mathcal{D} \subset \mathbb{R}^{n_x}$ be a domain containing $x = 0$.

Definition A.1. The equilibrium point $x = 0$ of (A.1) is:

- *stable* if, for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \forall t \geq 0$$

- *unstable* if it is not stable.
- *asymptotically stable* if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

The following theorems define the type of stability for the equilibrium point $x = 0$ via a function $V(x)$ called Lyapunov function.

Theorem A.1. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0, \ \forall x \in \mathcal{D} - \{0\} \\ &\dot{V}(x) \leq 0, \ \ \forall x \in \mathcal{D}, \end{aligned} \tag{A.2}$$

then, the equilibrium point $x = 0$ is *stable*.

Theorem A.2. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0, \quad \forall x \in \mathcal{D} - \{0\} \\ &\dot{V}(x) < 0, \quad \forall x \in \mathcal{D} - \{0\}, \end{aligned} \tag{A.3}$$

then, the equilibrium point $x = 0$ is *asymptotically stable*.

Theorem A.3. Let $V : \mathbb{R}^{n_x} \to \mathbb{R}$ be a continuously differentiable function such that

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0, \ \forall x \neq 0 \\ &\|x\| \to \infty \Rightarrow V(x) \to \infty \\ &\dot{V}(x) < 0, \quad \forall x \neq 0, \end{aligned} \tag{A.4}$$

then, the equilibrium point $x = 0$ is *globally asymptotically stable*.

Remark A.1: Sufficient conditions are achieved with these theorems; it means that if a function $V(x)$ cannot be found such that the conditions are satisfied no conclusions can be drawn respect to the stability of the equilibrium point.

APPENDIX B. Matrix Inequalities

B.1. Convex set

A set $\mathbb{F}$ is convex if for any $x_1, x_2 \in \mathbb{F}$ and $0 \le \theta \le 1$ there holds

$$\theta x_1 + (1-\theta) x_2 \in \mathbb{F}. \tag{B.1}$$

A function $f : \mathbb{R}^m \to \mathbb{R}$ is called convex if f is a non-empty convex set and if for all $x_1, x_2 \in \mathbb{R}^m$ and $0 \le \theta \le 1$ there holds that

$$f\left(\theta x_1 + (1-\theta) x_2\right) \le \theta f(x_1) + (1-\theta) f(x_2). \tag{B.2}$$

B.2. Linear Matrix Inequalities

In this section is presented a brief review about LMI framework. For more details check (Boyd et al., 1994; Scherer and Weiland, 2000). The standard form for a Linear matrix inequality (LMI) is defined as:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i < 0, \tag{B.3}$$

where $x \in \mathbb{R}^m$ is an unknown vector containing the decision variables and $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i \in 0,1,\ldots,m$, are known constant matrices. The inequality $F(x) < 0$ means that all eigenvalues $\lambda\left(F(x)\right)$ are negative or the maximal eigenvalue $\lambda_{\max}\left(F(x)\right) < 0$.

Standard problems

The three common convex or quasi-convex optimization problems for the analysis and controller design of systems are introduced below.

1. The *feasibility problem* (FP) is to find a solution $x \in \mathbb{R}^m$ such that the following LMI holds

$$F(x) < 0, \tag{B.4}$$

   where x is the vector of scalar decision variables. If the maximal eigenvalue $\lambda_{\max}(F(x))$ is negative the LMI problem is feasible, otherwise it is unfeasible.

2. The *eigenvalue problem* (EVP) is to minimize a linear function of the decision variables subject to LMI constraints, i.e.:

$$\begin{cases} \min \quad c^T x \\ s.t. \quad F(x) < 0, \end{cases} \tag{B.5}$$

   where x is the vector of scalar decision variables and c a vector of appropriate dimensions.

3. The *generalized eigenvalue problem* (GEVP) is to find a solution $x \in \mathbb{R}^m$ to the following minimization problem

$$\begin{cases} \min \quad \lambda \\ s.t. \quad A(x) < \lambda B(x),\ B(x) > 0,\ C(x) > 0 \end{cases} \tag{B.6}$$

   where x is the vector of scalar decision variables, λ is a scalar, and matrices $A(x)$, $B(x)$, and $C(x)$ are symmetric and affine in x.

B.3. Bilinear Matrix Inequalities

A bilinear matrix inequality (BMI) has the next structure (VanAntwerp and Braatz, 2000):

$$F(x,y) = F_0 + \sum_{i=1}^{m} x_i F_i + \sum_{j=1}^{n} y_j G_j + \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j H_{ij} < 0, \tag{B.7}$$

where F_i, G_j, and H_{ij} $i \in \{0,1,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$ are symmetric matrices, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are decision variables vectors. The BMI (B.7) is not convex at same time in x and y.

Therefore, a BMI as in (B.7) is only an LMI in x or y for a fixed y or x respectively. The main drawback of BMIs is that they are not optimally solvable because existing methods may lead to local minima.

B.4. Matrix properties

Property B.1 (Congruence transformation). Consider $\mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{n\times n}$ and a full row rank matrix $\mathcal{X} \in \mathbb{R}^{m\times n}$. The following expression holds:

$$\mathcal{Q} < 0 \Leftrightarrow \mathcal{X}\mathcal{Q}\mathcal{X}^T < 0; \qquad \mathcal{Q} > 0 \Leftrightarrow \mathcal{X}\mathcal{Q}\mathcal{X}^T > 0. \tag{B.8}$$

Property B.2 (Schur complement). Let $\mathcal{M} = \mathcal{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, with M_{11} and M_{22} square matrices of appropriate dimensions. Then:

$$\mathcal{M} < 0 \ \Leftrightarrow \ \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \Leftrightarrow \ \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}$$

Property B.3 (Peaucelle transformation). Let $\mathcal{A}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{Q}$ be matrices of proper size. The following inequalities are equivalent:

$$\mathcal{A}^T\mathcal{P} + \mathcal{P}^T\mathcal{A} + \mathcal{Q} < 0 \tag{B.9}$$

$$\exists \mathcal{R}, \mathcal{L} : \begin{bmatrix} \mathcal{A}^T\mathcal{L}^T + \mathcal{L}\mathcal{A} + \mathcal{Q} & \mathcal{P}^T - \mathcal{L} + \mathcal{A}^T\mathcal{R} \\ \mathcal{P} - \mathcal{L}^T + \mathcal{R}^T\mathcal{A} & -\mathcal{R} - \mathcal{R}^T \end{bmatrix} < 0. \tag{B.10}$$

Property B.4. For $y \in \mathbb{R}^n$ and $\beta > 0$, the following equivalence holds:

$$y^T y - \beta < 0 \Leftrightarrow yy^T - \beta I < 0 \tag{B.11}$$

Property B.5. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Q} = \mathcal{Q}^T > 0$ matrices of appropriate dimension, the following holds:

$$\mathcal{X}^T\mathcal{Y} + \mathcal{Y}^T\mathcal{X} \le \mathcal{X}^T\mathcal{Q}\mathcal{X} + \mathcal{Y}^T\mathcal{Q}^{-1}\mathcal{Y}, \tag{B.12}$$

if $\mathcal{Q} = \alpha I$, with $\alpha > 0$ a scalar, then (B.12) yields

$$\mathcal{X}^T\mathcal{Y} + \mathcal{Y}^T\mathcal{X} \le \alpha\mathcal{X}^T\mathcal{X} + \alpha^{-1}\mathcal{Y}^T\mathcal{Y}. \tag{B.13}$$

Property B.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be matrices of appropriate dimension; thus, if $|\lambda| < \beta$ then

$$\mathcal{Y} \pm \beta\mathcal{X} \le 0 \Rightarrow \mathcal{Y} + \lambda\mathcal{X} \le 0. \tag{B.14}$$

Property B.7: Given $\mathcal{P}=\mathcal{P}^T>0$, then

$$\mathcal{Q}^T\mathcal{P}^{-1}\mathcal{Q}\geq\mathcal{Q}^T+\mathcal{Q}-\mathcal{P}\,. \tag{B.15}$$

Lemma B.1. **(Finsler's lemma)** (de Oliveira and Skelton, 2001). Let $x\in\mathbb{R}^n$, $\mathcal{Q}=\mathcal{Q}^T\in\mathbb{R}^{n\times n}$, and $\mathcal{R}\in\mathbb{R}^{m\times n}$ such that $rank\left(\mathcal{R}\right)<n$; the following statements are equivalent:

$$\begin{aligned}&\text{a)}\;\; x^T\mathcal{Q}\,x<0,\quad \forall x\in\left\{x\in\mathbb{R}^n : x\neq 0, \mathcal{R}x=0\right\}.\\&\text{b)}\;\; \exists\mathcal{X}\in\mathbb{R}^{n\times m} : \mathcal{Q}+\mathcal{X}\mathcal{R}+\mathcal{R}^T\mathcal{X}^T<0.\end{aligned} \tag{B.16}$$

Lemma B.2 (Differential Mean Value Theorem): Let $f\left(z\right):\mathbb{R}^n\rightarrow\mathbb{R}$ and $a,b\in\mathbb{R}^n$. If $f\left(z\right)$ is a differentiable function on $\left[a,b\right]$ then, there exists a vector $c\in\mathbb{R}^n$ with $c\in\left]a,b\right[$ such that:

$$f\left(b\right)-f\left(a\right)=\frac{\partial f\left(c\right)}{\partial z}\left(b-a\right)=\nabla f\left(c\right)\left(b-a\right). \tag{B.17}$$

APPENDIX C. Sum relaxations

It is well-known that TS-LMI framework usually leads to inequalities containing convex sums. These inequalities include the membership functions (MFs) which contain nonlinear functions which must be removed in order to get LMI conditions. Expressing conditions in terms of LMIs is not a trivial task. Therefore, different sum relaxations have been developed in this sense. Some sum relaxations which are applied along the thesis are presented.

Consider a general case of inequalities with multiple nested convex sums:

$$\Upsilon_{\underbrace{hh\cdots h}_{q}} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_q=1}^{r} h_{i_1} h_{i_2} \dots h_{i_q} \Upsilon_{i_1 i_2 \dots i_q} < 0\,, \tag{C.1}$$

where $\Upsilon_{i_1 i_2 \dots i_q}$, $i_1, i_2, \dots, i_q \in \{1, 2, \dots, r\}$ are symmetric matrices of appropriate size.

The sign of such expressions should be established via LMIs, which implies that the MFs therein should be adequately dealt with: conditions thus obtained will be therefore only sufficient. This is why selecting a proper way to perform this task is important to reduce conservatism.

For single sums ($q = 1$ in (C.1)), the following lemma arises.

Lemma C.1 (Wang et al., 1996). Let Υ_{i_1}, $i_1 \in \{1, 2, \cdots, r\}$ be matrices of the same size. Condition (C.1) is verified for $q = 1$ if the following LMIs hold:

$$\Upsilon_{i_1} < 0\,, \forall i_1 \in \{1, 2, \dots, r\}. \tag{C.2}$$

When double sums are involved ($q=2$ in (C.1)), usually for controller/observer design, two schemes have been proposed in the literature. The following lemmas present these sum relaxations.

Lemma C.2 (Wang et al., 1996). Let $\Upsilon_{i_1 i_2}$, $i_1, i_2 \in \{1, 2, \cdots, r\}$ be matrices of the same size. Condition (C.1) is verified for $q=2$ if the following LMIs hold:

$$\begin{aligned} &\Upsilon_{i_1 i_1} < 0, \quad \forall i_1 \in \{1, 2, \cdots, r\} \\ &\Upsilon_{i_1 i_2} + \Upsilon_{i_2 i_1} < 0, \quad i_1, i_2 \in \{1, 2, \cdots, r\},\ i_1 < i_2. \end{aligned} \tag{C.3}$$

Lemma C.3 (Tuan et al., 2001)**.** Let $\Upsilon_{i_1 i_2}$, $i_1, i_2 \in \{1, 2, \cdots, r\}$ be matrices of the same size. Condition (C.1) is verified for $q=2$ if the following LMIs hold:

$$\begin{aligned} &\Upsilon_{i_1 i_1} < 0, \quad \forall i_1 \in \{1, 2, \cdots, r\} \\ &\frac{2}{r-1}\Upsilon_{i_1 i_1} + \Upsilon_{i_1 i_2} + \Upsilon_{i_2 i_1} < 0, \quad i_1, i_2 \in \{1, 2, \cdots, r\},\ i_1 \neq i_2. \end{aligned} \tag{C.4}$$

Should more than two nested convex sums be involved, a generalization of the sum relaxation in lemma C.2 will be used.

Lemma C.4 (Sala and Ariño, 2007)**.** Let $\Upsilon_{i_1 \ldots i_q}$, $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, r\}$ be matrices of the same size and $\rho(i_1, i_2, \cdots, i_q)$ be the set of all permutations of the indexes $i_1, i_2, \cdots, i_q$. Condition (C.1) is verified the following LMIs hold:

$$\sum_{i_1, i_2, \cdots, i_q \in \rho(i_1, i_2, \cdots, i_q)} \Upsilon_{i_1, i_2, \cdots, i_q} < 0, \quad i_1, i_2, \cdots, i_q \in \{1, 2, \ldots, r\}. \tag{C.5}$$

Remark C.1. There exists other relaxation sums which include extra slack matrices on the LMIs (Kim and Lee, 2000; Liu and Zhang, 2003; Fang et al., 2006; Sala and Ariño, 2007). However, it may leads to computational burden. In this thesis, sum relaxations without extra slack matrices are adopted.

When different MFs are involved in the convex sums, for instance: 1) use of a Lyapunov functional (section 3.2.5); 2) stability or controller design for descriptor models (section 3.3), it is necessary to make some adaptation of previous sum relaxations.

Inequality with convex sums for case 1) has the following structure:

$$\Upsilon_{hh h^- s} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j_1=1}^{r}\sum_{k_1=1}^{r} h_{i_1} h_{i_2} h^-_{j_1} s_{k_1} \Upsilon_{i_1 i_2}^{j_1 k_1} < 0, \tag{C.6}$$

where $\Upsilon_{i_1 i_2}^{j_1 k_1}$, $i_1, i_2, j_1, k_1 \in \{1, 2, \ldots, r\}$ are symmetric matrices of appropriate dimensions.

Inequality with convex sums for case 2) has the next form:

$$\Upsilon_{hhv} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{k_1=1}^{r_e} h_{i_1} h_{i_2} v_{k_1} \Upsilon_{i_1 i_2}^{k_1} < 0, \tag{C.7}$$

where $\Upsilon_{i_1 i_2}^{k_1}$, $i_1, i_2 \in \{1, 2, \ldots, r\}$, $k_1 \in \{1, 2, \ldots, r_e\}$ are symmetric matrices of appropriate size.

For those cases the following extensions of Lemma C. 3 are given.

Lemma C.5. Let $\Upsilon_{i_1 i_2}^{j_1 k_1}$, $i_1, i_2, j_1, k_1 \in \{1, 2, \ldots, r\}$ be matrices of the same size. Condition (C.6) is satisfied if the following LMIs hold:

$$1. \quad \begin{aligned} &\Upsilon_{i_1 i_1}^{j_1 k_1} < 0, \;\; \forall (i_1, j_1, k_1) \in \{1, 2, \cdots, r\} \\ &\frac{2}{r-1} \Upsilon_{i_1 i_1}^{j_1 k_1} + \Upsilon_{i_1 i_2}^{j_1 k_1} + \Upsilon_{i_2 i_1}^{j_1 k_1} < 0, \;\; i_1, i_2 \in \{1, 2, \cdots, r\},\; i_1 \neq i_2, \forall (j_1, k_1) \in \{1, 2, \cdots, r\}. \end{aligned} \tag{C.8}$$

Lemma C.6. Let $\Upsilon_{i_1 i_2}^{k_1}$, $i_1, i_2 \in \{1, 2, \ldots, r\}$, $k_1 \in \{1, 2, \ldots, r_e\}$ be matrices of the same size. Condition (C.7) is satisfied if the following LMIs hold:

$$\begin{aligned} &\Upsilon_{i_1 i_1}^{k_1} < 0, \;\; \forall i_1 \in \{1, 2, \cdots, r\}, \;\; \forall k_1 \in \{1, 2, \cdots, r_e\} \\ &\frac{2}{r-1} \Upsilon_{i_1 i_1}^{k_1} + \Upsilon_{i_1 i_2}^{k_1} + \Upsilon_{i_2 i_1}^{k_1} < 0, \;\; i_1, i_2 \in \{1, \cdots, r\},\; i_1 \neq i_2, \forall k_1 \in \{1, 2, \cdots, r_e\}. \end{aligned} \tag{C.9}$$

Résumé étendu

Chapitre 1: Introduction.

Dans les dernières décennies, un grand nombre de travaux sur l'analyse non linéaire et le design ont été menées sur la base de la représentation exacte de polytopique des systèmes non linéaires aussi connu comme modèles Takagi-Sugeno (TS) (Takagi et Sugeno, 1985). Une représentation TS peut être obtenu à partir d'un modèle non linéaire par l'intermédiaire de linéarisation dans plusieurs points d'intérêt (Tanaka et Wang, 2001), ou par l'approche du secteur non-linéarité, d'abord proposée en (Kawamoto et al., 1992) et prolongé par (Ohtake et al ., 2001; Taniguchi et al., 2001). Perte d'information est le principal problème des techniques de linéarisation, qui donnent une approximation du système non linéaire, un problème qui ne figure pas dans l'approche de la non-linéarité du secteur. Par conséquent, l'approche du secteur non-linéarité a été généralement appliquée afin d'obtenir un modèle TS. Un modèle TS est composé d'un ensemble de modèles linéaires mélangés avec fonctions d'appartenance (MFs en anglais) qui contiennent les non-linéarités du modèle et détiennent la propriété de somme convexe (Tanaka et Wang, 2001). Il ya plusieurs raisons derrière l'intérêt croissant sur l'analyse de la stabilité et de la conception contrôleur et d'observateur des systèmes non linéaires via des modèles TS: (a) ils peuvent représenter exactement une grande famille de modèles non linéaires dans un ensemble compact de l'espace de l'Etat par l'intermédiaire de l'approche du secteur non-linéarité; (b) sa structure convexe sur la base de fonctions d'appartenance permet aux méthodes linéaires d'être «facilement» imité par la méthode directe de Lyapunov (Tanaka et Wang, 2001); (c) des manipulations appropriées tout à fait avec compensation parallèle distribuée (PDC en anglais) comme une loi de commande conduisent généralement à des conditions sous la forme d'inégalités matricielles linéaires (LMI en anglais), qui sont efficacement résolu par des techniques d'optimisation convexe (Boyd et al., 1994; Scherer et Weiland, 2000).

Le cadre TS-LMI a été à l'origine basé sur une fonction de Lyapunov quadratique (QLF en anglais) de telle sorte que plusieurs résultats sur l'analyse de la stabilité ainsi que la conception contrôleur et observateur ont été largement abordés (Tanaka et Sugeno, 1992; Wang et al., 1996; Tanaka et al ., 1998; Patton et al., 1998; Tanaka et Wang, 2001; Bergsten et al., 2002; Ichalal et al., 2008;. Lendek et al., 2010b). Néanmoins, des conditions LMI ainsi obtenues, bien que simple, étaient seulement suffisante, ce qui signifie que le conservatisme est introduit dans les solutions, à savoir, si les conditions LMI sont irréalisables, il ne signifie pas que le problème d'origine n'a pas de solution. Trois sources indépendantes de prudence ont été identifiés: (1) la façon dont les MFs sont retirés de sommes convexes imbriqués pour obtenir des conditions LMI suffisantes, (2) le type de fonction de Lyapunov, et (3) la non-unicité de la construction de modèle TS. Par conséquent, un effort énorme a été consacrée à atteindre nécessité ou, au moins, se détendre suffisance afin de jeter une grande famille de problèmes dans le cadre TS-LMI (Sala et al., 2005; Feng et al., 2005). Un grand nombre de résultats sont disponibles qui couvrent partiellement une ou plusieurs de ces trois problèmes.

Pour (1), l'obtention expressions LMI de sommes convexes imbriqués a été abordé via les propriétés de la matrice (Tanaka et Sugeno, 1992; Tuan et al., 2001), par l'intermédiaire de conditions asymptotiquement suffisantes et nécessaires dépendant de paramètres (Sala et Ariño, 2007), en utilisant l'approche de triangulation pour aller à conditions asymptotiquement exactes (Kruszewski et al., 2009), et en ajoutant des variables d'écart (Kim et Lee, 2000; Liu et Zhang, 2003).

Pour (2), une importante littérature est maintenant disponible qui exploitent l'utilisation de plus générale fonction de Lyapunov comme par morceaux (PWLF en anglais) (Johansson et al., 1999; Feng et al., 2005; Campos et al., 2013), floue (FLF en anglais, aussi connu dans la littérature comme non-quadratique ou base-dépendante) (Tanaka et al., 2003; Guerra et Vermeiren, 2004), et la intégrale de ligne (LILF en anglais) (Rhee et Won, 2006; Mozelli et al. , 2009). Ces fonctions générales de Lyapunov partagent les mêmes MFs que le modèle TS. L'utilisation de PWLF se sont révélés être particulièrement difficile à traiter depuis des généralisations par morceaux de la fonction de Lyapunov quadratique exigent des conditions supplémentaires pour garantir sa continuité (Johansson et al., 1999). Dans le cadre-temps continu, les fonctions de Lyapunov floues ont pas rencontré le développement du domaine en temps discret (Guerra et Vermeiren, 2004; Guerra et al., 2009; Ding, 2010; Zou et Li, 2011). Cette asymétrie est due au fait que les dérivées temporelles des MFs apparaissent dans l'analyse et ne peuvent pas être facilement exprimés comme un problème convexe; en outre, elle conduit à une analyse locale qui peut créer des boucles algébriques lorsque la conception du contrôleur est concerné (Blanco et al., 2001). Parmi les travaux sur l'approche non quadratique local, deux directions peuvent être trouvées: celles qui assume simplement limites connues a priori des dérivés de la MFs (Tanaka et al., 2003; Bernal et al., 2006; Mozelli et al., 2009; Zhang et Xie, 2011; Lee et al., 2012; Yoneyama, 2013), et celles qui réécrit la dérivée de la MFs à obtenir des limites plus structurés (Guerra et Bernal, 2009; Bernal et Guerra, 2010 ; Pan et al., 2012; Jaadari et al., 2012; Lee et Kim, 2014). FLFs devraient être utilisés pour obtenir des conditions globaux, des solutions de rechange en intégrale de ligne peuvent être considérées. Dans le travail séminal (Rhee et Won, 2006), les auteurs ont montré comment les fonctions de Lyapunov par intégrale de ligne peuvent être utilisés pour éviter les dérivés de MFs au prix d'imposer des structures restrictives pour garantir la ligne intégrante d'être indépendant du trajectoires; en outre, cette approche conduit à inégalités matricielles bilinéaires (BMI en anglais) pour la conception de contrôleur; par conséquent, ils ne sont pas de façon optimale solvable parce que les méthodes existantes peuvent conduire à des minima locaux.

Pour (3) autres modèles convexes en plus de ceux TS ont été utilisées: polynôme (Tanaka et al., 2009 ; Sala, 2009) et le descripteur (Taniguchi et al., 1999). La structure de descripteur paru dans (Luenberger, 1977) avec le principal intérêt de décrire familles non linéaires des systèmes d'une manière plus naturelle que l'espace d'un état standard, généralement des systèmes mécaniques (Luenberger, 1977; Dai, 1989). Modèle de descripteur de TS est similaire à la version standard, la différence est que le descripteur a généralement deux familles de MFs, un pour le côté gauche et l'autre pour le côté droit. Dans (Taniguchi et al., 1999), la stabilité et la stabilisation des systèmes de descripteurs flous ont été présentés sous un régime quadratique; ce travail tire parti de la structure de descripteur de réduire le nombre de contraintes LMI, réduisant ainsi la charge de calcul. De meilleurs résultats pour la stabilisation ainsi que la conception du contrôleur robuste ont été présentés dans (Guerra et al., 2007) et (Bouarar et al., 2010), respectivement.

Le problème de l'estimation de l'état des systèmes dynamiques est l'un des principaux sujets de la théorie du contrôle et a donc été abondamment traité dans la littérature; son importance se pose clairement du fait que la loi de commande dépend souvent de variables d'état qui peuvent ne pas être disponibles en raison des capteurs coût élevé, inexistence ou impraticabilité. Estimation d'état à la fois pour les systèmes linéaires et non linéaires ont été proposées il ya longtemps (Luenberger, 1971; Thau, 1973); les plus récents travaux sur le sujet sont: techniques basées sur la mode glissant (Efimov et Fridman, 2011), approche à grand gain non linéaire (Khalil et Praly, 2014; Prasov et Khalil, 2013), approche de gain variable dans le temps (Farza et al. , 2014), et des extensions tenu en compte entrées inconnues sont également disponibles (Barbot et al., 2009; Bejarano et al., 2014). Conception d'observateur pour les modèles TS peut être séparé en deux catégories: la première considère que les MFs dépendent de variables mesurées (Tanaka et al., 1998; Patton et al., 1998; Teixeira et al., 2003; Akhenak et al., 2007; Lendek et al., 2010a); le second suppose que les MFs sont également formés par des variables non mesurées (Bergsten et al., 2001; Bergsten et al., 2002; Ichalal et al., 2007; Yoneyama, 2009; Lendek et al., 2010a; Ichalal et al., 2011; Ichalal et al., 2012). Pour la première classe, les résultats obtenus dans le cadre quadratique ressemblent à la dualité caractéristique observateurs/contrôleurs de systèmes linéaires. Pour la seconde classe, une façon de faire face à cette classe de variables non mesurées est de tenir compte des conditions supplémentaires en utilisant des constantes de Lipschitz classiquement comme dans (Ichalal et al., 2007). Une autre façon est d'utiliser le Théorème de valeur moyenne différentiel (DMVT en anglais) comme dans (Ichalal et al., 2011; Ichalal et al., 2012).

Il est difficile d'extraire ce que sont les vrais résultats importants; il existe un besoin pour converger vers des méthodes "utiles". Les idées suivies par cette thèse, quels qu'ils soient (élargissement de la fonction de Lyapunov, la loi de commande, les sommes imbriqués, le vecteur d'état), tentent de réduire le conservatisme des anciens résultats. Par exemple, pourquoi est-il pertinent d'introduire des lois de contrôle dont la complexité peut conduire à des conditions moins conservatrices si il ya déjà conditions nécessaires et suffisantes asymptotiques (ANS en anglais) pour la conception de contrôleur PDC basé sur fonction de Lyapunov quadratique? La raison réside dans le fait que les conditions ANS ont été obtenus uniquement pour les sommations convexes (Sala et Ariño, 2007 ; Kruszewski et al., 2009) dont la charge de calcul atteint très rapidement une taille prohibitif pour les solveurs actuels; ainsi, les approches en préservant les caractéristiques asymptotiques tout en atteignant des solutions où les conditions ANS ne peuvent pas, méritent d'être explorées. L'exemple suivant illustre les limitations des méthodes ANS. Une représentation TS, $\dot{x}=\sum_{i=1}^{r}h_i\left(z\right)\left(A_i x+B_i u\right)$, ainsi que d'une loi de commande PDC, $u=\sum_{i=1}^{r}h_i\left(z\right)Fx_i$, sont considérés dans l'analyse en vertu d'une fonction de Lyapunov quadratique (plus de détails dans le chapitre 2). Cet exemple est construit comme suit (Delmotte et al., 2007): envisager une représentation TS avec 2 modèles

$$A_1 = \begin{bmatrix} 0.5 & 0 \\ 1 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & -1 \\ -1 & 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \tag{R.1}$$

qui est prouvée pour être stabilisable via une loi de commande PDC ordinaire et une fonction de Lyapunov quadratique. Complexité de la représentation peut être introduite artificiellement en ajoutant des modèles à l'intérieur du polytope originale. Les matrices ainsi obtenues sont équidistants, à savoir: $\left(A_{\delta_k}, B_{\delta_k}\right)$, $\delta_k = \dfrac{k}{r-1}$ avec $k \in \{1,2,\ldots,r-2\}$ correspond à:

$$A_{1+\delta_k} = \left(1-\delta_k\right)A_1 + \delta_k A_2,\ B_{1+\delta_k} = \left(1-\delta_k\right)B_1 + \delta_k B_2. \tag{R.2}$$

Ainsi, le stabilisabilité quadratique par une loi de commande PDC est garanti indépendamment du nombre de modèles. Conditions de stabilisation en (Sala et Ariño, 2007) utiliser la propriété de Polya (Scherer, 2006) introduisant des sommes supplémentaires dans le problème initial $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Upsilon_{ij} < 0$ avec $\Upsilon_{ij} = A_i P + B_i F_j + (*)$, à savoir:

$$\left(\sum_{i=1}^{r} h_i(z)\right)^d \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\Upsilon_{ij} < 0, \tag{R.3}$$

où d représente le *paramètre de complexité* pour (R.3). Notez que s'il existe une solution au problème initial, il *doit exister* une valeur suffisamment grande de d telle sorte que le problème (R.3) est faisable. Théorème 5 dans (Sala et Ariño, 2007) ajoute également quelques variables supplémentaires assouplir les conditions pour une valeur fixe de d. Malgré sa simplicité, les conditions dans Théorème 5 (Sala et Ariño, 2007) avec $r = 10$ et $d = 2$, conduisent solveurs LMI à l'échec. Dans ce cas, le nombre de conditions LMI et variables de décision scalaires sont 41123 et 772, respectivement. Cet exemple montre que, parfois, des problèmes très simples ne peuvent pas être résolus, même si les conditions ANS sont disponibles. Ainsi, il est important d'explorer des alternatives qui offrent des conditions plus souples, par exemple, les fonctions de Lyapunov non-quadratique.

Cette thèse propose de nouveaux systèmes de contrôle et d'observation pour représentations TS des systèmes non linéaires à temps continu tels que les conditions plus souples sont atteints. Les problèmes considérés sont:

- conception de contrôleur de retour d'état.

- conception d'observateur.

Les stratégies sont appliquées pour les modèles TS dans une forme standard et descripteur. Tous les développements sont basés sur la méthode directe de Lyapunov tels que les conditions LMI (ou ceux paramétrés) sont obtenus.

Chapitre 2: Préliminaires sur les modèles de Takagi-Sugeno.

Ce chapitre présente la base de la modélisation sous une structure convexe de systèmes non linéaires (modèle TS) ainsi que les principaux résultats sur l'analyse de la stabilité et de la conception contrôleur et d'observateur pour ce genre de modèles dans les cadres quadratique et non quadratique. En outre, certains résultats sur les modèles TS dans une forme de descripteur sont fournis mettront en lumière les avantages de ce système par rapport à la modélisation standard.

Résultats sur l'analyse de la stabilité et de la conception du contrôleur en vertu d'un cadre quadratique et non quadratique à travers LMI ont été présentés en soulignant les principales contributions et les inconvénients de ces approches. En outre, le problème de l'estimation de l'état pour les systèmes dynamiques a été traitée à la fois pour les variables de prémisse mesurées et non mesurées. En outre, descripteur TS régime de modèle et diverses propositions à ce sujet ont été résumées. Quelques exemples ont été donnés pour clarifier les concepts et les approches.

Les problèmes suivants ont été abordés dans les chapitres suivants afin de fournir une proposition de solution pour y faire face:

- Malgré le fait que conditions asymptotiquement nécessaires et suffisantes (ANS) sont fournies dans la littérature, la forte demande de ressources informatiques ainsi que le conservatisme associé au choix du candidat de fonction de Lyapunov ou le régime de la loi de contrôle particulier sont encore des problèmes ouverts.
- Même avec l'utilisation de fonctions de Lyapunov non quadratique pour réduire prudence des conditions suffisantes parce que le régime quadratique, dans le cas continu se pose la nécessité de gérer les dérivés de la MFs dont difficile de trouver des conditions globaux au problème sur conception de contrôleur.
- Conception d’observateur pour les modèles TS sous variables non mesurées qui n’est pas facile de jeter comme un problème convexe.

Chapitre 3 : Conception de contrôleur pour les modèles Takagi-Sugeno.

Ce chapitre présente quelques contributions sur la conception de contrôleur de retour d'état pour les systèmes non linéaires à temps continu. Les méthodologies sont basées sur des représentations TS exactes des configurations non linéaires à l'étude; les formes standard ainsi que descripteurs sont adressées.

La première partie est sur les modèles TS standard. Les schémas de conception de contrôleur sont basées sur: 1) une fonction de Lyapunov quadratique (QLF) (Tanaka et Wang, 2001); 2) une fonction de Lyapunov floue (FLF) (Tanaka et al., 2003); 3) une fonction de Lyapunov intégrale de ligne (LILF) (Rhee et Won, 2006); 4) un nouveaux fonctionnelle de Lyapunov non quadratique (NQLF). Schémas 1) et 2) incorporer un système de détente somme basée sur plusieurs sommes convexes que celui de (Sala et Ariño, 2007); dans ce cas, les améliorations proviennent de transformations de matrice tels que ceux en (Shaked, 2001), (de Oliveira et Skelton, 2001), et (Peaucelle et al., 2000). Extensions à la conception de la performance $H\infty$ sont faites.

La deuxième partie concerne les modèles de descripteurs TS. Deux stratégies sont proposées: 1) dans le cadre quadratique, des conditions basées sur une loi de commande général et la transformation de matrice à (Peaucelle et al., 2000); une extension de rejet perturbation $H\infty$ est présenté; 2) une extension de l'approche non-quadratique (Rhee et Won, 2006) pour les systèmes de second ordre, qui utilise une fonction de Lyapunov par intégrale de ligne (LILF), une loi de commande non-PDC, et le lemme du Finsler; cette stratégie offre conditions LMI dépendant des paramètres à la place de contraintes BMI. Des améliorations sont présentées via des exemples illustratifs longs du chapitre.

Plusieurs méthodologies pour la conception de contrôleur de retour d'état, les deux les formes standard ainsi que descripteur TS représentations des systèmes non linéaires à temps continu, ont été présentés. Les stratégies proposées sont principalement basées sur des transformations de matrices telles que le lemme du Finsler ainsi qu'une variété de fonctions de Lyapunov tels que floue et l'intégrale de ligne. De plus, une nouvelle fonction de Lyapunov a été proposée pour être utilisé à la place de fonctions de Lyapunov. Les améliorations sur la conception du contrôleur via QLF et une loi de commande imbriquée multiple ont été atteintes en préservant les caractéristiques asymptotiques; ces améliorations apportent une réduction de la charge de calcul (pour aider les solveurs numériques) ainsi que l'inclusion des résultats précédents (théorème de Polya) comme un cas particulier. En outre, ces améliorations ont été étendues en utilisant des fonctions de Lyapunov floues telles qu'une réduction significative sur le conservatisme est obtenu. En outre, le problème de rejet de perturbation a été résolu. Toutes les stratégies présentées produits de plus grands ensembles de faisabilité, en préservant leur expression LMI jusqu'à paramètre dépendances qui peuvent être traitées par programmation linéaire ou recherche espacées logarithmique.

Chapitre 4 : Conception d'observateur pour les modèles Takagi-Sugeno.

Ce chapitre fournit des contributions sur l'estimation d'état pour les systèmes non linéaires à temps continu via les modèles TS; ils sont divisés en deux parties: le cas particulier où les vecteurs de prémisses sont basés sur les variables mesurées et le cas général où les vecteurs de prémisses peuvent être basés sur des variables non mesurées.

La première partie propose des systèmes de conception d'observateurs de plus en plus détendue sur la base de: (1) une transformation de matrices «Tustin-like» apparu dans (Shaked, 2001) (2) le Lemme du Finsler (Jaadari et al., 2012) et (3) une transformation basés sur (Peaucelle et al., 2000). Tous ces régimes sera prolongée pour incorporer plusieurs sommes convexes imbriqués (Márquez et al., 2013). En outre, les extensions directes à rejet de perturbation $H\infty$ sont développées.

La deuxième partie est plus difficile car il fait face le cas général, à savoir, une structure d'observateur qui facilite la manipulation des fonctions d'appartenance qui dépendent de variables non mesurées $h_i(\hat{z})$. Anciens résultats sur le sujet des variables non mesurées considèrent l'erreur de fonction d'appartenance $h_i(z)-h_i(\hat{z})$ tout à fait avec des constantes de Lipschitz classiques (Bergsten et al., 2001; Ichalal et al., 2007); ce ne sera pas l'approche présentes considéré. L'approche en (Ichalal et al., 2011; Ichalal et al., 2012) est poursuivi dans cette thèse; la conception d'observateur est basée sur le théorème de la valeur moyenne différentiel. Ainsi, les conditions LMI assurant la convergence asymptotique de l'erreur d'estimation d'état à zéro sont obtenues; ces conditions sont étendues à la conception de la performance $H\infty$.

Nouvelles approches pour la conception d'observateur des modèles non linéaires de temps continu qui surmontent la mise à jour des résultats dans l'état de l'art, ont été rapportés. Une première série de résultats se concentre sur la conception d'observateur basé dans les prémisses mesurables: profitant d'une réécriture convexe du modèle (forme TS) ainsi que de plusieurs transformations de matrice tels que le lemme du Finsler, l'observateur est découplée de leur fonction de Lyapunov correspondant; en outre, la proposition de découplage permet l'introduction progressive de meilleurs résultats grâce à une structure convexe imbriqué. Une deuxième série de solutions dans ce chapitre examine le cas prémisse non mesurées: le problème d'estimation d'état est exprimée comment une convexe en utilisant d'une fonction de Lyapunov quadratique et le théorème de la valeur moyenne; la structure ainsi obtenue permet l'introduction de variables d'écart. Tant le cas des prémisses mesuré et non mesurée comprennent des extensions sur les perturbations rejet $H\infty$, où de meilleurs résultats que ceux déjà disponibles dans la littérature sont signalés.

Chapitre 5 : Conclusions et perspectives.

Ce chapitre donne quelques conclusions générales sur les résultats présentés dans cette thèse ainsi que les directions possibles pour la recherche future.

Cette thèse a été dirigée sur l'analyse des systèmes non linéaires à temps continu via des modèles de Takagi-Sugeno. Les TS représentations suivantes ont été considérés dans cette thèse: 1) modèle TS forme standard (R.4) et 2) modèle TS forme descripteur (R.5).

Modèle TS forme standard: $\dot{x}=\sum_{i=1}^{r}h_i\left(z\right)\left(A_ix+B_iu\right),\ y=\sum_{i=1}^{r}h_i\left(z\right)\left(C_ix\right).$ (R.4)

Modèle TS forme descripteur: $\sum_{k=1}^{r}v_k\left(z\right)E_k\dot{x}=\sum_{i=1}^{r}h_i\left(z\right)\left(A_ix+B_iu\right),\ y=\sum_{i=1}^{r}h_i\left(z\right)\left(C_ix\right).$(R.5)

Les problèmes suivants ont été abordés:

- conception de contrôleur de retour d'état.
- conception d'observateur.

Le but de toutes nouvelles approches a été porté pour réduire la prudence sur les conditions de résultats antérieurs dans le cadre TS-LMI.

Nouveaux schémas pour la *conception de contrôleur de retour d'état* ont été développés pour les représentations TS (R.4) et (R.5). Pour le cas standard, ces systèmes qui sont fondés sur certaines propriétés de la matrice peut être divisé en deux parties: 1) par l'intermédiaire de fonctions de Lyapunov quadratique et 2) par l'intermédiaire des fonctions Lyapunov non quadratique. Pour la première partie, le schéma proposé incorpore une relaxation basée sur de multiples sommes convexes tels que les caractéristiques asymptotiques sont gardés. Pour la deuxième partie, les régimes basés sur floue (FLF) ou des fonctions de Lyapunov par intégrale de ligne (LILF) ont été présentés. Les conditions ainsi obtenus sont local ou global dans un ensemble compact de l'espace de l'Etat pour FLF et LILF, respectivement. Pour le problème de stabilisation basée sur une LILF, la structure convexe reste jusqu'à ce que les systèmes de second ordre. En outre, une nouvelle fonctionnelle de Lyapunov non quadratique (NQLF) a été proposée qui conduit à des conditions moins conservatrices. Pour le cas de descripteur, les régimes ont été développés sur la base de QLF ainsi que LILF. En outre, le problème de rejet de perturbation $H\infty$ a été abordé. Toutes ces stratégies donnent des conditions plus souples pour les œuvres précédentes.

Dans le cas de la conception d'observateur seul modèle TS standard (R.4) a été envisagée. Deux directions ont été discutées: 1) vecteurs de prémisses sont basées sur les variables mesurées et 2) Les vecteurs de prémisses sont basées sur variables non mesurées ou les deux. Dans la première direction conditions de plus en plus détendu via une fonction de Lyapunov quadratique et plusieurs sommes imbriqués ont été obtenus. Certaines solutions de rechange par l'intermédiaire des transformations de matrice ont été données. Pour la deuxième direction, un nouveau schéma basé sur le théorème de valeur moyenne a été abordée. La faisabilité de cette approche régler surmonte les résultats des méthodes antérieures sur cette direction. Les conditions LMI obtenues assurent la convergence asymptotique à zéro pour l'erreur d'estimation d'état. Une extension de ces résultats à la conception de la performance $H\infty$ a été présenté. Les contributions développés pour la conception observateur/contrôleur sont résumés dans les tableaux suivants.

Tableau 5.1. Contributions sur la conception du contrôleur

Conception de contrôleur	QLF	FLF	LILF	NQLF
Modèle TS standard	Approches généralisées: - Lemmes techniques - Multiples sommes convexes	Approche généralisées	LMIs pour les systèmes de second ordre	LMI sous un nouveau fonctionnel de Lyapunov
Modèle TS descriptor	LMI dépendants de paramètres: - Loi de commande étendue - Finsler lemme	-	LMIs pour les systèmes de second ordre	-

Tableau 5.2. Contributions sur la conception d'observateur

Conception d'observateur	Variables prémisses mesurées	Variables prémisses non mesurées
QLF	Approches généralisées: - Lemmes techniques - Multiples sommes convexes	LMI basé sur le théorème de valeur moyenne

Certaines directions pour la recherche future sont :

- Stabilisation avec multiple sommes convexes imbriquées.

Lorsque la loi de commande suivante a été adoptée:

$$u = F_{hh\cdots h} H_{hh\cdots h}^{-1} x, \tag{R.6}$$

où $F_{\underbrace{hh\cdots h}_{q}}$ et $H_{\underbrace{hh\cdots h}_{q}}$ sont des matrices convexes et « q » est le nombre de sommes convexes.

Quelle que soit l'approche utilisée tout à fait avec la loi de commande complexes (R.6), il peut conduire à des problèmes à la intention des implémentation parce que plusieurs multiplications et inversion de la matrice doivent être effectuées sur les applications en temps réel, en particulier dans les représentations TS non linéaires avec un nombre élevé de règles . Cependant, certaines pistes peuvent être explorées. Une fois que nous obtenons une fonction de Lyapunov de stabilisation, il est possible de: 1) applique des régressions sur la loi de commande pour obtenir une plus simple (approximation); 2) construire une carte numérique (peut utiliser beaucoup de mémoire); 3) utiliser la fonction de Lyapunov obtenus et recherche pour un contrôleur simple par LMI (peut conduire à une impossibilité si le problème est sensible) ou via un contrôle optimal (peut conduire à des gains élevés).

- Les fonctions de Lyapunov non quadratique.

Un résultat intéressant pour la stabilisation qui contourne la manipulation des les dérivés des MFs, sont basées sur une fonction de Lyapunov par intégrale de ligne qui conduit à des conditions globales au sein de l'ensemble compact de l'espace d'état (Rhee et Won, 2006). Cette approche a été améliorée au moins pour les systèmes de second ordre conduisant à des conditions LMI à la place de BMI. Néanmoins, si les systèmes d'ordre supérieur sont considérés, alors le problème ne peut pas être traité comme un problème convexe. Afin de comprendre les difficultés qui se posent sur le problème de contrôle associé avec $V(x) = 2\int_{\Gamma(0,x)} f(\psi)d\psi$, $f(\psi) = P_h^{-1}x$, il suffit de considérer le cas d'un modèle TS des troisième ordre et 2 règles. Dans ce cas, la fonction de Lyapunov doit vérifier les conditions de la trajectoire indépendante (voir chapitre 3), ce qui signifie que la matrice de la fonction de Lyapunov a la forme suivante, et sont les variables inconnues:

$$P_h^{-1} = \begin{bmatrix} d_\omega & q_{12} & q_{13} \\ q_{12} & d_2 & q_{23} \\ q_{13} & q_{23} & d_3 \end{bmatrix}, \tag{R.7}$$

Bien entendu, le problème vient du fait que les contraintes LMI sont écrites conformément à P_h, soit avec $P_h = \frac{1}{\left|P_h^{-1}\right|}\begin{bmatrix} d_2 d_3 - q_{23}^2 & q_{13}q_{23} - q_{12}d_3 & q_{12}q_{23} - q_{13}d_2 \\ q_{13}q_{23} - q_{12}d_3 & d_{\omega 1}d_3 - q_{13}^2 & q_{12}q_{13} - q_{23}d_{\omega 1} \\ q_{12}q_{23} - q_{13}d_2 & q_{12}q_{13} - q_{23}d_{\omega 1} & d_{\omega 1}d_2 - q_{12}^2 \end{bmatrix}$. Malgré le fait que

$\left|P_h^{-1}\right|$ peut être facilement retiré des contraintes LMI (de la même façon que pour le cas de 2ème ordre), la matrice de jouxtent présente des conditions BMI pour le cas de 3ème ordre et des conditions polynômes pour ordre supérieur et up-to-maintenant aucune transformation est disponibles pour rendre le problème LMI.

Par conséquent, les pistes qui peuvent être poursuivis besoin de quelques opérations de matrice supplémentaires, tels que le lemme de Finsler ou l'utilisation de descripteur redondance. Un autre point de vue pourrait être de résoudre une sorte d'algorithme de 2 étape. La première serait la solution à envisager le problème général $A_h P_h + B_h F_h + (*) < 0$ avec P_h maintien de la structure de trajectoire indépendant. Notez que la dualité est pas satisfait, cela ne garantit pas que le «vrai» problème impliquant la structure de trajectoire indépendant de P_h^{-1} sera satisfait. Ce point est illustré par l'intermédiaire de contre-exemple dans (Guelton et al., 2010). Si une solution est trouvée, puis utilisez les gains obtenus F_h dans un problème de stabilité unique de la boucle fermée qui est un problème contraintes LMI comme dans (Rhee et Won, 2006).

- Fonctionnelle de Lyapunov Non quadratique.

Une nouvelle approche par un *fonctionnelle de Lyapunov non quadratique* (NQLF) a été proposés qui donne également les conditions globaux sans imposer de fortes restrictions sur la structure des MFs comme intégrales de ligne indépendants du parcours font et sans aucune limitation de l'ordre du système. Le NQLF a la structure suivante:

$$V(x) = x^T P_s^{-1} x = x^T \left(\sum_{i=1}^{r} s_i(z(t)) P_i \right)^{-1} x, \tag{R.8}$$

avec $P_i = P_i^T > 0$, et

$$s_i(z(t)) = \frac{1}{\alpha} \int_{t-\alpha}^{t} h_i(z(\tau)) d\tau \geq 0, \ \alpha > 0. \tag{R.9}$$

Suivant la même généralisation qui peut être trouvé dans (Ding, 2010, Lee et al., 2011, Estrada-Manzo et al., 2015), avec des fonctions de Lyapunov à sommes multiples, (R.8) pourraient être facilement généralisés à fonctionnel tel que:

$$V(x) = x^T P_{\underbrace{s\ldots s}_{q}}^{-1} x = x^T \left(\sum_{i_1=1}^{r} \ldots \sum_{i_q=1}^{r} s_{i_1}(z) \ldots s_{i_q}(z) P_{i_1 \ldots i_q} \right)^{-1} x, \tag{R.10}$$

où tous les $s_{i_j}(z)$, $j \in \{1,2,\ldots,q\}$ correspond à la définition (R.9), ou encore définir des retards différents α_j, $j \in \{1,2,\ldots,q\}$:

$$s_{i_j}(z(t)) = \frac{1}{\alpha_j}\int_{t-\alpha_j}^{t} h_{i_j}(z(\tau))d\tau \geq 0,\ \alpha_j > 0. \tag{R.11}$$

Par conséquent, ces extensions aussi directement de sens pour systèmes de temporisation TS depuis la fonctionnelle de Lyapunov comporte déjà implicitement retards qui pourraient être facilement à manipuler.

Considérez le candidat fonctionnelle suivante classique Lyapunov-Krasovskii pour les systèmes temps-retard (Gu et al., 2003):

$$V = \underbrace{x^T P x}_{V_1} + \underbrace{\int_{t-\tau}^{t} x^T(s) S x(s) ds}_{V_2} + \underbrace{\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta}_{V_3}, \tag{R.12}$$

où τ est le temps de retard. L'idée est de substituer le terme V_1 de (R.12) par le NQLF dans (R.8) ou son extension (R.10), et essayer d'obtenir des conditions LMI que l'amélioration anciens résultats.

- Conception d'observateur.

Toutes les approches présentées dans cette thèse pour l'estimation de l'État assument une fonction de Lyapunov quadratique pour les deux: vecteur des prémisses mesurées et non mesurées. Par conséquent, les fonctions de Lyapunov non quadratique ou fonctionnelle peuvent être appliquées dans l'analyse afin de réduire le degré de suffisance des conditions LMI. Néanmoins, ce problème est loin d'être facile. Dans le «simple» cas la fonction de Lyapunov écrit:

$$V(e) = e^T P_h e = x^T \sum_{i=1}^{r} h_i(z(t)) P_i x, \tag{R.13}$$

avec l'erreur d'état dynamique $e = x - \hat{x}$ sous la forme:

$$\dot{e} = \left(A_h - P_h^{-1} K_h C_h\right) e. \tag{R.14}$$

Puis naturellement la dérivée de la fonction de Lyapunov comprendra $P_{\dot{h}}$ qui écrit (voir chapitre 2):

$$P_{\dot{h}} = \sum_{i=1}^{r} \sum_{k=1}^{p} h_i \frac{\partial w_0^k}{\partial z_k} \left(P_{g_1(i,k)} - P_{g_2(i,k)} \right) \dot{z}_k \,, \tag{R.15}$$

avec $g_1(i,k) = \lfloor (i-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (i-1) \bmod 2^{p-k}$ et $g_2(i,k) = g_1(i,k) + 2^{p-k}$, $\lfloor \bullet \rfloor$ se dresse pour la fonction de plancher. Par conséquent, la dérivée de la fonction de Lyapunov inclut explicitement $\dot{z}_k(t)$, ce qui signifie une partie de $\dot{x}(t)$. Ainsi, il est impossible de faire face directement à l'observation d'un système instable, que pour faire face à (R.15), il sera nécessaire d'avoir délimité $\left\| \dot{z}_k(t) \right\|$. Ce point peut être considéré comme une limitation sérieuse.

En autre, les résultats présentés dans ce travail à propos de l'estimation d'état sont développés pour le modèle TS standard, par conséquent, il serait intéressant de les prolonger pour un descripteur représentation TS.

- Les applications temps réel.

Tous les résultats présentés dans cette thèse sont dans un cadre théorique. Néanmoins, il est également important de tester les différentes approches prévues à ce travail pour les systèmes physiques différents et de comparer avec d'autres stratégies comme contrôleur heuristique floue (sans modèle) ou méthodologies classiques sur le contrôle non linéaire. Certains systèmes physiques universitaires qui sont disponibles à ITSON qui pourraient être utilisés sont: le rail inversé de pendule, le pendule Furuta, double rotor MIMO, réservoirs interconnectés, et de la grue 3D.